

Linear Algebra 1: Tutorial 9

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Definition. Let n be a positive integer, and let π be a permutation in S_n . An inversion in π is an ordered pair (p, q) of integers in $\{1, \dots, n\}$ such that $p < q$ and $\pi(p) > \pi(q)$.¹

Problem 1 from HW#6. Let n be a positive integer. Prove that for any permutation $\pi \in S_n$, we have that $\text{sgn}(\pi) = (-1)^r$, where r is the number of inversions (see the definition above) in π .

Hint: Proceed by induction on the number of inversions. In the induction step, show that if a permutation $\pi \in S_n$ has $k + 1$ inversions, then there exists a permutation π' and a transposition (ij) in S_n such that $\pi = (ij) \circ \pi'$ and π' has k inversions.²

Exercise 1. In this exercise, all entries are assumed to be in \mathbb{Z}_3 . Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 2 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 & 2 \end{bmatrix},$$

and vectors

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

¹For example, the permutation $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$ has four inversions, namely $(1, 4), (2, 3), (2, 4), (3, 4)$.

²For example, for $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$, we can take $\pi' = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, and we observe that $\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}}_{=\pi} = (34) \circ \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}}_{=\pi'}$, and that π has four inversions, whereas π' has three.

For the following, try to minimize the amount of row-reducing that you have to do.

- (a) Find $\text{rank}(A)$.
- (b) Find a basis for $\text{Col}(A)$.
- (c) Find a basis for $\text{Row}(A)$.
- (d) Find a basis for $\text{Nul}(A)$.
- (e) For $i = 1, 2$, determine whether \mathbf{x}_i belongs to $\text{Col}(A)$, and if so, express \mathbf{x}_i as a linear combination of the basis vectors for $\text{Col}(A)$ that you found in part (b).
- (f) For $i = 1, 2$, determine whether \mathbf{y}_i^T belongs to $\text{Row}(A)$, and if so, express \mathbf{y}_i^T as a linear combination of the basis vectors for $\text{Row}(A)$ that you found in part (c).
- (g) For $i = 1, 2$, determine whether \mathbf{y}_i belongs to $\text{Nul}(A)$.

Exercise 2. Determine which (if any) of the following are bases for \mathbb{R}^5 . You should be able to answer all the questions without doing any messy calculations.

$$\bullet \mathcal{B}_1 := \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \end{bmatrix}, \begin{bmatrix} 1 \\ \sqrt{3} \\ 3 \\ 3\sqrt{3} \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ \sqrt{7} \\ 7 \\ 7\sqrt{7} \\ 49 \end{bmatrix} \right\}$$

$$\bullet \mathcal{B}_2 := \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \end{bmatrix}, \begin{bmatrix} 1 \\ \sqrt{3} \\ 3 \\ 3\sqrt{3} \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ \sqrt{7} \\ 7 \\ 7\sqrt{7} \\ 49 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 9 \\ 27 \\ 81 \end{bmatrix} \right\}$$

$$\bullet \mathcal{B}_3 := \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \end{bmatrix}, \begin{bmatrix} 1 \\ \sqrt{3} \\ 3 \\ 3\sqrt{3} \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ \sqrt{7} \\ 7 \\ 7\sqrt{7} \\ 49 \end{bmatrix}, \begin{bmatrix} 1 \\ \sqrt{2} \\ 2 \\ 2\sqrt{2} \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \end{bmatrix} \right\}$$

$$\bullet \mathcal{B}_4 := \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\bullet \mathcal{B}_5 := \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\bullet \mathcal{B}_6 := \left\{ \begin{bmatrix} 1 \\ 3 \\ 6 \\ 9 \\ 12 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\bullet \mathcal{B}_7 := \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\bullet \mathcal{B}_8 := \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 8 \\ 9 \\ 10 \end{bmatrix}, \begin{bmatrix} 11 \\ 12 \\ 13 \\ 14 \\ 15 \end{bmatrix} \right\}$$

Exercise 3. Let \mathbb{F} be an infinite field,³ and let $\text{Func}(\mathbb{F}, \mathbb{F})$ be the set of all functions from \mathbb{F} to \mathbb{F} . Then $\text{Func}(\mathbb{F}, \mathbb{F})$ is a vector space over \mathbb{F} (you may assume this). Prove that $\text{Func}(\mathbb{F}, \mathbb{F})$ is infinite-dimensional.

Hint: Consider functions in $\text{Func}(\mathbb{F}, \mathbb{F})$ that are zero-valued almost everywhere, but “jump” at one value in the domain \mathbb{F} .

³This means that $|\mathbb{F}|$ is infinite, i.e. \mathbb{F} has infinitely many elements.