Linear Algebra 1: Tutorial 7

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Definition. *Let n be a positive integer, and let π be a permutation in S_n. An* inversion *in π is an ordered pair (p, q) of integers in $\{1, \dots, n\}$ such that $p < q$ and $\pi(p) > \pi(q)$.*[1]

Exercise 1. *By Problem 1 from HW#6, for any permutation $\pi \in S_n$, we have that $sgn(\pi) = (-1)^r$, where r is the number of inversions in π. For each of the following permutations in S_7, find the list of all inversions of the permutation, and then using that, find the sign of the permutation.*

(a) $\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 3 & 6 & 4 & 5 & 7 \end{pmatrix}$;

(b) $\pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 2 & 4 & 5 & 7 & 6 \end{pmatrix}$.

Theorem 2.7 from Lecture Notes 6. *Let V be a vector space over a field $\mathbb{F}$, and let $U \subseteq V$. Then U is a subspace of V if and only if the following three conditions are satisfied:*

(i) $\mathbf{0} \in U$;

(ii) U is closed under vector addition, that is, for all $\mathbf{u}, \mathbf{v} \in U$, we have that $\mathbf{u} + \mathbf{v} \in U$;

(iii) U is closed under scalar multiplication, that is, for all $\mathbf{u} \in U$ and $\alpha \in \mathbb{F}$, we have that $\alpha\mathbf{u} \in U$.

[1] For example, the permutation $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$ has four inversions, namely $(1, 4), (2, 3), (2, 4), (3, 4)$.

Exercise 2. *Let $\mathbb{F}$ be a field, and let $\mathbb{P}_{\mathbb{F}}$ be the vector space of all polynomials (in variable x) with coefficients in $\mathbb{F}$. Determine whether the following are subspaces of $\mathbb{P}_{\mathbb{F}}$. Does the answer depend on what $\mathbb{F}$ is?*

(a) $P_1 := \{ax^2 \mid a \in \mathbb{F}\}$;

(b) $P_2 := \{x^2 + a \mid a \in \mathbb{F}\}$.

Exercise 3. *Given a field $\mathbb{F}$, we say that matrices $A, B \in \mathbb{F}^{n\times n}$* commute *if $AB = BA$.*

(a) Is the set of matrices $A \in \mathbb{R}^{2\times 2}$ that commute with the matrix $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ a subspace of $\mathbb{R}^{2\times 2}$?

(b) Might the answer to part (a) change if B were some other matrix in $\mathbb{R}^{2\times 2}$?

(c) What if, instead of matrices in $\mathbb{R}^{2\times 2}$, we considered matrices in $\mathbb{F}^{2\times 2}$, for some other field $\mathbb{F}$? And what about matrices in $\mathbb{F}^{n\times n}$ for an arbitrary positive integer n?

Exercise 4. *Let U and W be vector spaces over a field $\mathbb{F}$. (Let us denote the zero vectors in U and W by $\mathbf{0}_U$ and $\mathbf{0}_W$, respectively.) Now consider*

$$U \times W \quad := \quad \{(\mathbf{u}, \mathbf{w}) \mid \mathbf{u} \in U,\ \mathbf{w} \in W\},$$

with vector addition and scalar multiplication defined in the usual way, i.e. as follows:

- *for all $(\mathbf{u}_1, \mathbf{w}_1), (\mathbf{u}_2, \mathbf{w}_2) \in U \times W$, we set $(\mathbf{u}_1, \mathbf{w}_1) + (\mathbf{u}_2, \mathbf{w}_2) = (\mathbf{u}_1 + \mathbf{u}_2, \mathbf{w}_1 + \mathbf{w}_2)$;*
- *for all $(\mathbf{u}, \mathbf{w}) \in U \times W$ and $\alpha \in \mathbb{F}$, we set $\alpha(\mathbf{u}, \mathbf{w}) = (\alpha\mathbf{u}, \alpha\mathbf{w})$.*

Explain why $U \times W$ is a vector space over $\mathbb{F}$. Explain, moreover, why Theorem 2.7 from Lecture Notes 6 ***cannot*** *be used to prove that $U \times V$ is a vector space.*

Exercise 5. *Let $\mathbb{F}$ be a finite field,[2] with additive identity 0 and multiplicative identity 1. Prove that there exists some positive integer n such that $\underbrace{1 + \cdots + 1}_{n \text{ times}} = 0$.*

[2] Technically, this means that $(\mathbb{F}, +, \cdot)$ is a field and that $\mathbb{F}$ is a finite set.

> ***Hint:*** *Consider the sequence* $\{\underbrace{1+\cdots+1}_{k \text{ times}}\}_{k=1}^{\infty}$, *i.e. the sequence* $1,\ 1+1,\ 1+1+1,\ 1+1+1+1, \ldots$, *and prove that some two elements of this sequence are equal. Now what?*

Definition. *The* characteristic *of a finite field* $\mathbb{F}$ *with additive identity* 0 *and multiplicative identity* 1, *denoted by char*$(\mathbb{F})$, *is the smallest positive integer* n *such that* $\underbrace{1+\cdots+1}_{n \text{ times}} = 0$.[3]

Exercise 6. *Let* $\mathbb{F}$ *be a finite field, with additive identity* 0 *and multiplicative identity* 1. *Prove that char*$(\mathbb{F})$ *is a prime number.*

> ***Hint:*** *If there exist integers* $p, q \geq 2$ *such that char*$(\mathbb{F}) = pq$, *then consider the product* $(\underbrace{1+\cdots+1}_{p \text{ times}})(\underbrace{1+\cdots+1}_{q \text{ times}})$.

[3] By Problem 5, this is well-defined.