Linear Algebra 1: Tutorial 6

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Winter 2022/2023

Problem 1 from HW#3. *Let $\mathbb{F}$ be a field,*[1] *and let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a function. Prove that the following are equivalent:*

(1) f is linear;

(2) for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{F}^m$ and all scalars $\alpha, \beta \in \mathbb{F}$, we have that

$$f(\alpha\mathbf{u} + \beta\mathbf{v}) \quad = \quad \alpha f(\mathbf{u}) + \beta f(\mathbf{v}).$$

Exercise 1 from Tutorial 5. *Let*

$$A \quad = \quad \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} \qquad \text{and} \qquad B \quad = \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

with entries understood to be in $\mathbb{Z}_2$. Solve the equation $AX = B$. How many solutions does the equation $AX = B$ have?

Exercise 2 from Tutorial 5. *Let*

$$A \quad = \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \text{array} \qquad B \quad = \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

with entries understood to be in $\mathbb{R}$. Solve the equation $AX = B$. How many solutions does the equation $AX = B$ have?

Exercise 3 from Tutorial 5. *Let*

$$A \quad = \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad B \quad = \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 2 & 0 & 2 \end{bmatrix}$$

with entries understood to be in $\mathbb{Z}_3$. Solve the equation $XA = B$. How many solutions does the equation $XA = B$ have?

[1] For now, you may assume that $\mathbb{F}$ is either $\mathbb{R}$, or $\mathbb{C}$, or $\mathbb{Z}_p$ (for some prime p).

Exercise 1. *Let* $\mathbb{F}$ *be a field, and let* $A \in \mathbb{F}^{n\times m}$ *be a matrix. Let* $\left[\begin{array}{c|c} U & B \end{array}\right] = RREF(\left[\begin{array}{c|c} A & I_n \end{array}\right])$. *Explain why all the following hold:*

(a) $U = RREF(A)$;

(b) B *is invertible;*

(c) $U = BA$.

> ***Hint:*** *Elementary matrices.*

Exercise 2. *Let*

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

with entries understood to be in $\mathbb{Z}_3$. *Express* A *in the form* $A = BU$, *where* $U = RREF(A)$ *and* $B \in \mathbb{Z}_3^{2\times 2}$ *is an invertible matrix.*

> ***Hint:*** *Use Exercise 1.*

Exercise 3. *Let* $\mathbb{F}$ *be a field, let* $A_1, A_2 \in \mathbb{F}^{n\times m}$. *Prove that* $A_1 \sim A_2$ *(i.e.* A_1 *and* A_2 *are row equivalent) if and only if there exists an invertible matrix* $B \in \mathbb{F}^{n\times n}$ *such that* $A_2 = BA_1$.