Linear Algebra 1: Tutorial 2

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Exercise 1. *Consider any system of n linear equations in m variables, with coefficients in some field $\mathbb{F}$. (For now, you may assume that $\mathbb{F}$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Z}_p$ for some prime number p.) For each of the following three possibilities,*

1. $m < n$
2. $m = n$
3. $m > n$

determine which of the following is possible:

(a) the system is inconsistent (i.e. has no solutions);

(b) the system has a unique solution;

(c) the system has more than one solution.

If an outcome is impossible, explain why. If it is possible, give a concrete example of a linear system satisfying it (try to make your system as simple as possible).

> ***Hint:*** *Consider RREF of the augmented matrix of the system. What can you say about the pivot columns and free variables?*

Exercise 2. *Solve the following system of linear equations, with coefficients in $\mathbb{R}$.*

$$\begin{array}{rcrcrcrcrcr} 2x_1 & + & 4x_2 & + & x_3 & & & + & x_5 & = & 4 \\ 3x_1 & + & 6x_2 & + & x_3 & + & 2x_4 & + & 2x_5 & = & 11 \\ -x_1 & - & 2x_2 & & & - & 3x_4 & - & x_5 & = & 1 \end{array}$$

Exercise 3. *Solve the following system of linear equations, with coefficients in $\mathbb{Z}_2$.*

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 0 \\ x_1 & + & x_2 & & & = & 1 \\ x_1 & & & + & x_3 & = & 0 \\ & & x_2 & + & x_3 & = & 0 \end{array}$$

Exercise 4. *Solve the following system of linear equations, with coefficients in* $\mathbb{Z}_3$*.*

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & & & = & 2 \\ 2x_1 & + & x_2 & + & x_3 & = & 1 \\ x_1 & + & 2x_2 & + & x_3 & = & 0 \end{array}$$

Exercise 5. *Consider the following linear system, with coefficients in* $\mathbb{R}$*, and with a fixed parameter* $c \in \mathbb{R}$*.*

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & + & x_3 & = & 1 \\ 2x_1 & + & x_2 & + & 2x_3 & = & 0 \\ 4x_1 & - & x_2 & + & 4x_3 & = & c \end{array}$$

Determine for which values of the parameter c*, the system above is consistent. For such values of* c*, find the general solution of the system (in terms of* c*).*

Exercise 6. *Consider the following vectors in* $\mathbb{R}^5$*.*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}.$$

Determine which, if any, of the following vectors in $\mathbb{R}^5$ *belong to Span*$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$*.*

$$\mathbf{a} = \begin{bmatrix} 1 \\ 7 \\ -1 \\ 4 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ 5 \\ 0 \\ 4 \\ 0 \end{bmatrix}.$$

Exercise 7. *Solve the following system of linear equations, with coefficients in* $\mathbb{Z}_5$*.*

$$\begin{array}{rcrcrcl} 2x_1 & + & 3x_2 & + & 4x_3 & = & 0 \\ x_1 & + & 2x_2 & + & 3x_3 & = & 4 \\ 4x_1 & & & + & x_3 & = & 2 \end{array}$$