# Linear Algebra 1: Tutorial 11 (some solutions)

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Winter 2022/2023

**Problem 4 from HW#9.** *Let $U$ and $V$ be vector spaces over a field $\mathbb{F}$, and assume that $U$ is non-trivial (i.e. has at least one non-zero vector) and finite-dimensional. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be a linearly independent set in $U$, and let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$. Prove that there exists a linear transformation $f : U \to V$ such that $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_k) = \mathbf{v}_k$.*

*Solution.* Since $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a linearly independent set in the finite-dimensional vector space $U$, Proposition 1.10 from Lecture Notes 7 guarantees that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ can be extended to a basis of $U$, say $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_k, \dots, \mathbf{u}_n\}$. Now, fix any $\mathbf{v}_{k+1}, \dots, \mathbf{v}_n \in V$. Then by Theorem 2.2 from Lecture Notes 9, there exists a unique linear transformation $f : U \to V$ such that $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_n) = \mathbf{v}_n$. In particular, the linear transformation $f : U \to V$ satisfies $f(\mathbf{u}_1) = \mathbf{v}_1, \dots, f(\mathbf{u}_k) = \mathbf{v}_k$, which is what we needed. □

**Exercise 1.** *Consider the following polynomials in $\mathbb{P}^4_{\mathbb{Z}_3}$:*

- $p_1(x) = x^4 + 2$;
- $p_2(x) = x^3 + x^2$;
- $p_3(x) = x^4 + x^3 + x^2 + 2$;
- $p_4(x) = 2x^4 + x^3 + x^2 + 1$;
- $p_5(x) = 2x + 1$.

*Determine whether there exists a linear function $f : \mathbb{P}^4_{\mathbb{Z}_3} \to \mathbb{Z}^2_3$ such that*

- $f(p_1(x)) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$;
- $f(p_2(x)) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$;
- $f(p_3(x)) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$;

- $f(p_4(x)) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$;

- $f(p_5(x)) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

*Solution.* We know from Example 4.2 from Lecture Notes 9 that

$$\mathcal{B} \;:=\; \{p_1(x), p_2(x), p_5(x)\}$$

is a basis for $\mathrm{Span}(p_1(x), \ldots, p_5(x))$, and that

- $p_3(x) = p_1(x) + p_2(x)$;

- $p_4(x) = 2p_1(x) + p_2(x)$.

Since $\mathcal{B}$ is a basis for $\mathrm{Span}(p_1(x), \ldots, p_5(x))$, we know that $\mathcal{B}$ is a linearly independent set in $\mathbb{P}^4_{\mathbb{Z}_3}$. Therefore,[1] there exists a linear transformation $f : \mathbb{P}^4_{\mathbb{Z}_3} \to \mathbb{Z}_3^2$ that satisfies

- $f(p_1(x)) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$;

- $f(p_2(x)) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$;

- $f(p_5(x)) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

But any such linear transformation $f$ must satisfy

$$\begin{aligned} f(p_3(x)) &= f(p_1(x) + p_2(x)) \\ &= f(p_1(x)) + f(p_2(x)) \qquad \text{because } f \text{ is linear} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &\neq \begin{bmatrix} 0 \\ 2 \end{bmatrix}. \end{aligned}$$

This proves that there is no linear transformation with the desired properties (i.e. properties from the statement of the problem). $\square$

[1] By Problem 4 from HW#9, whose solution is given above.