Linear Algebra

Irena Penev

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1 Solving matrix equations of the form $AX = B$

Example 1.1. *Let*

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & 1 & -2 \\ 0 & 1 & 0 & 3 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 5 & 3 \\ -3 & 1 \\ 3 & 0 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Solve the equation $AX = B$.[1] *How many solutions does the equation $AX = B$ have?*

Solution #1. Set $X = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$.[2] Then $AX = \begin{bmatrix} A\mathbf{x}_1 & A\mathbf{x}_2 \end{bmatrix}$, and so the equation $AX = B$ is equivalent to

$$\begin{bmatrix} A\mathbf{x}_1 & A\mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}.$$

So, we need to solve two matrix-vector equations, namely

$$A\mathbf{x}_1 = \mathbf{b}_1 \quad \text{and} \quad A\mathbf{x}_2 = \mathbf{b}_2.$$

We solve these equations one by one.

First, we solve the equation $A\mathbf{x}_1 = \mathbf{b}_1$. We form the augmented matrix $\left[\begin{array}{c:c} A & \mathbf{b}_1 \end{array}\right]$ and row reduce to obtain the reduced row echelon form of this matrix.

$$\left[\begin{array}{c:c} A & \mathbf{b}_1 \end{array}\right] = \left[\begin{array}{cccc:c} 1 & 2 & 3 & 4 & 5 \\ -1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 0 & 3 & 3 \end{array}\right] \sim \left[\begin{array}{cccc:c} 1 & 0 & 0 & \frac{31}{4} & \frac{35}{4} \\ 0 & 1 & 0 & 3 & 3 \\ 0 & 0 & 1 & -\frac{13}{4} & -\frac{13}{4} \end{array}\right].$$

[1] Note that solutions of the matrix equation $AX = B$ are 4×2 real matrices.

[2] In other words, $\mathbf{x}_1$ is the first column of X, and $\mathbf{x}_2$ is the second column of X. Similarly, $\mathbf{b}_1$ is the first column of B, and $\mathbf{b}_2$ is the second column of B. Thus,

$$\mathbf{b}_1 = \begin{bmatrix} 5 \\ -3 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}.$$

We now read off the solutions for $\mathbf{x}_1$:

$$\mathbf{x}_1 \quad = \quad \begin{bmatrix} -\frac{31}{4}s + \frac{35}{4} \\ \\ -3s + 3 \\ \\ \frac{13}{4}s - \frac{13}{4} \\ \\ s \end{bmatrix}, \quad \text{with } s \in \mathbb{R}.$$

We now solve the equation $A\mathbf{x}_2 = \mathbf{b}_2$. We form the augmented matrix $\left[\begin{array}{c:c} A & \mathbf{b}_2 \end{array}\right]$ and we row reduce to obtain the reduced row echelon form of this matrix.

$$\left[\begin{array}{c:c} A & \mathbf{b}_1 \end{array}\right] \quad = \quad \left[\begin{array}{rrrr:r} 1 & 2 & 3 & 4 & 3 \\ -1 & 3 & 1 & -2 & 1 \\ 0 & 1 & 0 & 3 & 0 \end{array}\right] \quad \sim \quad \left[\begin{array}{rrrr:r} 1 & 0 & 0 & \frac{31}{4} & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -\frac{13}{4} & 1 \end{array}\right].$$

We now read off the solutions for $\mathbf{x}_2$:

$$\mathbf{x}_2 \quad = \quad \begin{bmatrix} -\frac{31}{4}t \\ \\ -3t \\ \\ \frac{13}{4}t + 1 \\ \\ t \end{bmatrix}, \quad \text{with } t \in \mathbb{R}.$$

We now read off the solution for $X = \left[\begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array}\right]$:

$$X \quad = \quad \begin{bmatrix} -\frac{31}{4}s + \frac{35}{4} & -\frac{31}{4}t \\ \\ -3s + 3 & -3t \\ \\ \frac{13}{4}s - \frac{13}{4} & \frac{13}{4}t + 1 \\ \\ s & t \end{bmatrix}, \quad \text{with } s, t \in \mathbb{R}.$$

There are two parameters (namely, s and t), and they can each take infinitely many values (because $\mathbb{R}$ is infinite). So, the equation $AX = B$ has infinitely many solutions. □

Remark: Note that the parameters (namely, s and t) from the solution above are different for the different columns! This is because we solved the equations $A\mathbf{x}_1 = \mathbf{b}_1$ and $A\mathbf{x}_2 = \mathbf{b}_2$ independently, and so the parameter that appears in $\mathbf{x}_1$ is independent from the one that appears in $\mathbf{x}_2$.

Solution #1 is correct, but rather inefficient. We had to solve a separate vector equation for each column of B,[3] and each of these vector equations involved forming an augmented matrix and finding its RREF. Luckily, we can do better by essentially solving these two vector equations simultaneously.

Solution #2. We first form the matrix $\left[\begin{array}{c:c} A & B \end{array}\right]$ and row reduce to find its reduced row echelon form.

$$\left[\begin{array}{c:c} A & B \end{array}\right] = \left[\begin{array}{rrrr:rr} 1 & 2 & 3 & 4 & 5 & 3 \\ -1 & 3 & 1 & -2 & -3 & 1 \\ 0 & 1 & 0 & 3 & 3 & 0 \end{array}\right]$$

After row reducing, we obtain the following matrix (the columns after the dotted line are color coded for easier reference):

$$RREF(\left[\begin{array}{c:c} A & B \end{array}\right]) = \left[\begin{array}{rrrr:rr} 1 & 0 & 0 & \frac{31}{4} & \frac{35}{4} & 0 \\ 0 & 1 & 0 & 3 & 3 & 0 \\ 0 & 0 & 1 & -\frac{13}{4} & -\frac{13}{4} & 1 \end{array}\right].$$

We now read off the columns of X one by one. We read off the first column of X by reading off the solutions of the matrix-vector equation encoded by the matrix obtained by taking the submatrix to the left of the dotted line, plus the first column to the right of the dotted line (i.e. the red column).[4] We read off the second column of X by reading off the solutions of the matrix-vector equation encoded by the matrix obtained by taking the submatrix to the left of the dotted line, plus the second column to the right of the dotted line (i.e. the blue column).[5] The solutions are as follows:[6]

$$X = \begin{bmatrix} -\frac{31}{4}s + \frac{35}{4} & -\frac{31}{4}t \\ -3s + 3 & -3t \\ \frac{13}{4}s - \frac{13}{4} & \frac{13}{4}t + 1 \\ s & t \end{bmatrix}, \quad \text{with } s, t \in \mathbb{R}$$

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[3] Since B has two columns, this translated into two vector equations. In general, if B has m columns, we get m vector equations.

[4] This is the matrix
$$\left[\begin{array}{rrrr:r} 1 & 0 & 0 & \frac{31}{4} & \frac{35}{4} \\ 0 & 1 & 0 & 3 & 3 \\ 0 & 0 & 1 & -\frac{13}{4} & -\frac{13}{4} \end{array}\right].$$

[5] This is the matrix
$$\left[\begin{array}{rrrr:r} 1 & 0 & 0 & \frac{31}{4} & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -\frac{13}{4} & 1 \end{array}\right].$$

[6] Remember to use different parameters for different columns!

There are two parameters (namely, s and t), and they can each take infinitely many values (because $\mathbb{R}$ is infinite). So, the equation $AX = B$ has infinitely many solutions. □

In general, suppose A is an $n \times m$ matrix and B is an $n \times p$ matrix, and we wish to solve the matrix equation $AX = B$.[7] We proceed as follows:

1. We form the $n \times (m + p)$ matrix $\left[\begin{array}{c:c} A & B \end{array}\right]$ and find its RREF.
2. We check if $RREF(\left[\begin{array}{c:c} A & B \end{array}\right])$ has a row of the form
$$\left[\begin{array}{ccc:ccc} 0 & \dots & 0 & * & \dots & * \end{array}\right],$$
where at least one of the $*$'s (after the dotted line) is non-zero.
   (a) If such a row exists, then the equation $AX = B$ is inconsistent (i.e. has no solutions).[8]
   (b) If no such row exits, then the equation $AX = B$ is consistent (i.e. has at least one solution). For each $k \in \{1, \dots, p\}$,[9] we read off the k-th column of X by focusing on the part of $RREF(\left[\begin{array}{c:c} A & B \end{array}\right])$ to the left of the dotted line, plus the k-th column of $RREF(\left[\begin{array}{c:c} A & B \end{array}\right])$ after the dotted line.
      - If there are any free variables, remember to use different letters for the parameters in different columns, as in the solution of Example 1.1.

Example 1.2. *Let*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 4 & 3 & 1 & 3 \\ 4 & 3 & 1 & 3 \\ 2 & 1 & 1 & 3 \\ 2 & 1 & 2 & 3 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Solve the equation $AX = B$.[10] *How many solutions does the equation $AX = B$ have?*

Solution. We first form the matrix

$$\left[\begin{array}{c:c} A & B \end{array}\right] = \left[\begin{array}{ccc:cccc} 1 & 1 & 1 & 4 & 3 & 1 & 3 \\ 1 & 1 & 1 & 4 & 3 & 1 & 3 \\ 1 & 2 & -1 & 2 & 1 & 1 & 3 \\ 1 & 2 & -1 & 2 & 1 & 2 & 3 \end{array}\right].$$

[7]Note that solutions of the matrix equation $AX = B$ are $m \times p$ matrices.

[8]Indeed, suppose we got a row of the form $\left[\begin{array}{ccc:ccc} 0 & \dots & 0 & * & \dots & * \end{array}\right]$, where the k-th $*$ after the dotted line is non-zero. Then there are no solutions for the k-th row of the matrix X, and therefore, there are no solutions for the matrix X, either.

[9]Remember: p is the number of columns of B, and therefore, the number of columns of X.

[10]Note that solutions of the matrix equation $AX = B$ are 3×4 real matrices.

After row reducing, we obtain

$$RREF([\ A \,\vdots\, B\ ]) \quad = \quad \left[\begin{array}{ccc:cccc} 1 & 0 & 3 & 6 & 5 & 0 & 3 \\ 0 & 1 & -2 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

By considering the third row of $RREF([\ A \,\vdots\, B\ ])$, we see that the equation $AX = B$ has no solutions. □

Example 1.3. *Let*

$$A \quad = \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad \textit{and} \qquad B \quad = \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_2$. Solve the equation $AX = B$. How many solutions does the equation $AX = B$ have?

Solution. We first form the matrix

$$[\ A \,\vdots\, B\ ] \quad = \quad \left[\begin{array}{cccc:ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right].$$

After row reducing, we obtain

$$RREF([\ A \,\vdots\, B\ ]) \quad = \quad \left[\begin{array}{cccc:ccc} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right].$$

We now read off the solutions:

$$X \quad = \quad \begin{bmatrix} t_1 + 1 & t_2 + 1 & t_3 \\ t_1 + 1 & t_2 & t_3 + 1 \\ t_1 + 1 & t_2 & t_3 \\ t_1 & t_2 & t_3 \end{bmatrix}, \quad \text{where } t_1, t_2, t_3 \in \mathbb{Z}_2.$$

There are three parameters (namely, t_1, t_2, t_3), and any one of them can take two values (because $|\mathbb{Z}_2| = 2$). So, the total number of solutions of the equation $AX = B$ is $2^3 = 8$. □

Example 1.4. *Let*

$$A \quad = \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix} \qquad \textit{and} \qquad B \quad = \quad \begin{bmatrix} 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 2 & 2 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_3$. Solve the equation $AX = B$. How many solutions does the equation $AX = B$ have?

Solution. We first form the matrix

$$\left[\begin{array}{c:c} A & B \end{array}\right] = \left[\begin{array}{ccc:cccc} 1 & 1 & 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 & 2 & 2 & 2 \end{array}\right].$$

After row reducing, we obtain

$$RREF(\left[\begin{array}{c:c} A & B \end{array}\right]) = \left[\begin{array}{ccc:cccc} 1 & 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right].$$

We now see that the equation $AX = B$ has a unique solution, namely,

$$X = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

□

2 Solving matrix equations of the form $XA = B$

Suppose we are asked to solve a matrix equation of the form $XA = B$. This equation is equivalent to the equation $(XA)^T = B^T$, which is, in turn, equivalent to $A^T X^T = B^T$. Using the methods from the previous section, we solve the equation $A^T X^T = B^T$ for X^T, and then we take the transpose of the solution(s) to obtain X.

Example 2.1. *Let*

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 1 & 1 & 0 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 5 & 5 & 1 & -2 \\ 4 & 3 & 1 & -1 \\ 2 & 4 & 0 & -2 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Solve the equation $XA = B$.[11] *How many solutions does the equation $XA = B$ have?*

Solution. Note that $XA = B$ if and only if $A^T X^T = B^T$. So, we first find all values of X^T satisfying $A^T X^T = B^T$, and then we take the transpose to obtain all the values of X that satisfy $XA = B$.

Clearly,

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B^T = \begin{bmatrix} 5 & 4 & 2 \\ 5 & 3 & 4 \\ 1 & 1 & 0 \\ -2 & -1 & -2 \end{bmatrix}.$$

[11] Note that solutions of the matrix equation $XA = B$ are 3×2 matrices with entries in $\mathbb{R}$.

We first form the matrix

$$\left[\begin{array}{c:c} A^T & B^T \end{array}\right] = \left[\begin{array}{cc:ccc} 1 & 3 & 5 & 4 & 2 \\ 2 & 1 & 5 & 3 & 4 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & 0 & -2 & -1 & -2 \end{array}\right].$$

By row reduction, we get that

$$RREF(\left[\begin{array}{c:c} A^T & B^T \end{array}\right]) = \left[\begin{array}{cc:ccc} 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Using the matrix above, we can solve for X^T. There is only one solution, namely:

$$X^T = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}.$$

Thus, the equation $XA = B$ has a unique solution, namely:

$$X = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}.$$

□

Example 2.2. *Let*

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_2$. Solve the equation $XA = B$.[12] *How many solutions does the equation $XA = B$ have?*

Solution. Note that $XA = B$ if and only if $A^T X^T = B^T$. So, we first find all values of X^T satisfying $A^T X^T = B^T$, and then we take the transpose to obtain all the values of X that satisfy $XA = B$.

Clearly,

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B^T = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

[12] Note that solutions of the matrix equation $XA = B$ are 2×3 matrices with entries in $\mathbb{Z}_2$.

We now form the matrix

$$\left[\begin{array}{c:c} A^T & B^T \end{array}\right] = \left[\begin{array}{ccc:cc} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

By row reducing, we obtain:

$$RREF([A^T \ B^T]) = \left[\begin{array}{ccc:cc} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

By considering the third row of $RREF([A^T \ B^T])$, we see that the equation $A^T X^T = B^T$ has no solutions. Consequently, $XA = B$ has no solutions. □

Example 2.3. *Let*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 \\ 3 & 0 & 3 & 0 & 3 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 2 & 3 & 2 & 3 & 2 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_5$*. Solve the equation* $XA = B$*.*[13] *How many solutions does the equation* $XA = B$ *have?*

Proof. Note that $XA = B$ if and only if $A^T X^T = B^T$. So, we first find all values of X^T satisfying $A^T X^T = B^T$, and then we take the transpose to obtain all the values of X that satisfy $XA = B$.

Clearly,

$$A^T = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad B^T = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 1 & 2 \\ 0 & 3 \\ 1 & 2 \end{bmatrix}.$$

We now form the matrix

$$\left[\begin{array}{c:c} A^T & B^T \end{array}\right] = \left[\begin{array}{ccc:cc} 1 & 1 & 3 & 1 & 2 \\ 1 & 2 & 0 & 0 & 3 \\ 1 & 1 & 3 & 1 & 2 \\ 1 & 2 & 0 & 0 & 3 \\ 1 & 1 & 3 & 1 & 2 \end{array}\right].$$

[13] Note that solutions of the matrix equation $XA = B$ are 2×3 matrices with entries in $\mathbb{Z}_5$.

By row reducing, we obtain:

$$RREF([\ A^T \mid B^T\ ]) \quad = \quad \left[\begin{array}{ccc|cc} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Using the matrix above, we can solve for X^T:

$$X^T \quad = \quad \begin{bmatrix} 4s+2 & 4t+1 \\ 3s+4 & 3t+1 \\ s & t \end{bmatrix}, \quad \text{where } s, t \in \mathbb{Z}_5.$$

Thus, the solutions of the matrix equation $XA = B$ are

$$X \quad = \quad \begin{bmatrix} 4s+2 & 3s+4 & s \\ 4t+1 & 3t+1 & t \end{bmatrix}, \quad \text{where } s, t \in \mathbb{Z}_5.$$

Since we have two parameters (namely, s and t), each of which can take five values (because $|\mathbb{Z}_5| = 5$), we see that the matrix equation $XA = B$ has 25 solutions. $\square$