Linear Algebra 1: Lecture 2

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1 Vectors

A *column vector*, or simply *vector*, is a matrix with just one column. Here are some examples of vectors (in this case, vector entries are real numbers):

$$\mathbf{a} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -13 \\ 0 \\ 0 \\ \pi \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Vectors are typically denoted by bold letters (e.g. $\mathbf{a}$, $\mathbf{u}$, or $\mathbf{x}$) or by letters with an arrow on top (e.g. $\vec{a}$, $\vec{u}$, or $\vec{x}$). The zero vector (i.e. vector $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$) is denoted by $\mathbf{0}$ or $\vec{0}$.

The set of all vectors with n entries, where all the entries are from some field $\mathbb{F}$, is denoted by $\mathbb{F}^n$.[1] So, in the example above, we have that $\mathbf{a} \in \mathbb{R}^2$, $\mathbf{b} \in \mathbb{R}^4$, and $\mathbf{c} \in \mathbb{R}^5$.

We can perform two operations on vectors: "vector addition" and "scalar multiplication," which are defined straightforwardly, as follows. Let $\mathbb{F}$ be some field.

- Given two vectors, say

  $$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

  in $\mathbb{F}^n$, we define the *sum* of $\mathbf{x}$ and $\mathbf{y}$ by

  $$\mathbf{x} + \mathbf{y} := \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix},$$

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[1] Recall that we have not defined fields yet. However, some examples of fields are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}_p$ (where p is a prime number).

where the sums $x_i + y_i$ (for $i = 1, \ldots, n$) are computed in the field $\mathbb{F}$.

- Given a vector

$$\mathbf{x} \;=\; \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

in $\mathbb{F}^n$ and a scalar $\alpha \in \mathbb{F}$, we define

$$\alpha\mathbf{x} \;:=\; \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix},$$

where the products $\alpha x_1, \ldots, \alpha x_n$ are computed in the field $\mathbb{F}$.

Example 1.1. *Consider the vectors*

$$\boldsymbol{x} \;=\; \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix} \qquad \textit{and} \qquad \boldsymbol{y} \;=\; \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

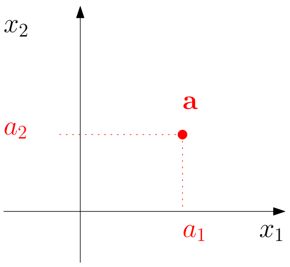
in $\mathbb{Z}_3^4$*. Then*

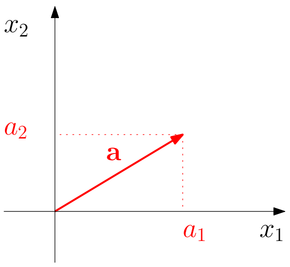
$$\mathbf{x}+\mathbf{y} \;=\; \begin{bmatrix} 0+1 \\ 1+0 \\ 2+2 \\ 2+1 \end{bmatrix} \;=\; \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad \textit{and} \qquad 2\mathbf{x} \;=\; \begin{bmatrix} 2\cdot 0 \\ 2\cdot 1 \\ 2\cdot 2 \\ 2\cdot 2 \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

(Remember: In $\mathbb{Z}_3$*, we have that* $2+2=1$*,* $2+1=0$*, and* $2\cdot 2 = 1$*.)*

1.1 Geometric interpretation of vectors in $\mathbb{R}^2$

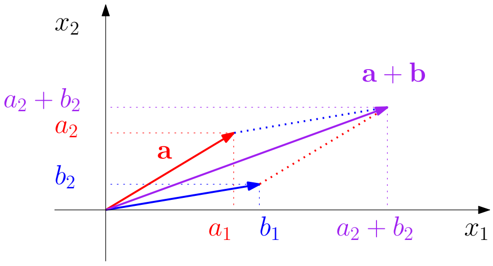
A vector $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ in $\mathbb{R}^2$ can be represented in the 2-dimensional Euclidean space either as a point (see the picture below, on the left) or as a line segment with an arrow starting at the origin (see the picture below, on the right).



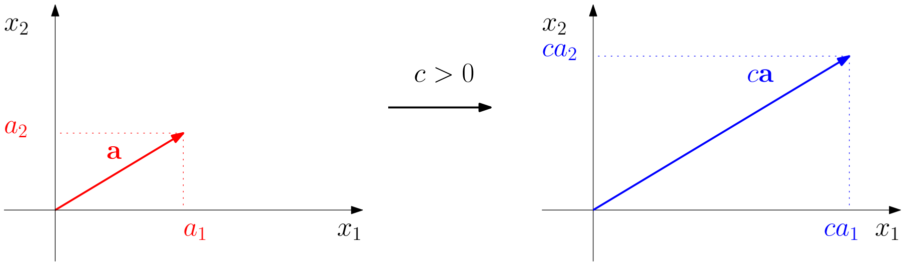


The zero vector $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is simply the origin.

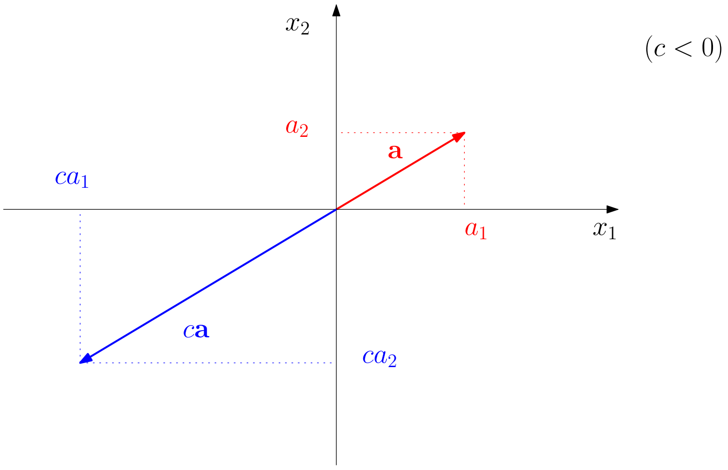
To add two vectors in $\mathbb{R}^2$, say $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ we apply the "parallelogram rule," as shown below.



Scalar multiplication can be interpreted as follows. Suppose we are given a vector $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and a scalar $c \in \mathbb{R}$. If $c > 0$, then $c\mathbf{a}$ is the vector that points in the same direction as $\mathbf{a}$, but whose length is scaled by c.



On the other hand, if $c < 0$, then $c\mathbf{a}$ is the vector that points in the opposite direction of $\mathbf{a}$, but whose length is scaled by $|c| = -c$.



If $c = 0$, then $c\mathbf{a} = \mathbf{0}$.

For vectors in $\mathbb{R}^3$, we have a similar geometric interpretation, only in the 3-dimensional Euclidean space.

1.2 Linear combinations of vectors

Suppose $\mathbb{F}$ is some field. A *linear combination* of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ in $\mathbb{F}^n$ is any sum of the form

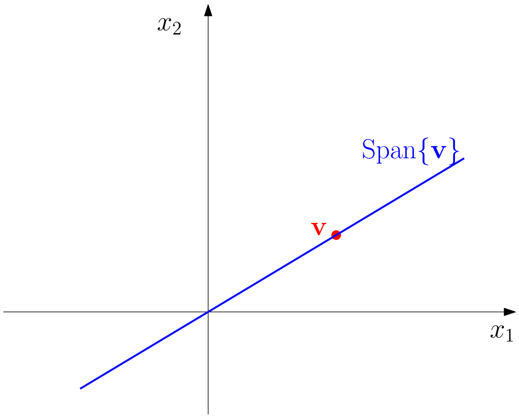
$$\sum_{i=1}^{m} \alpha_i \mathbf{v}_i \quad = \quad \alpha_1 \mathbf{v}_1 + \cdots + \alpha_m \mathbf{v}_m,$$

where $\alpha_1, \ldots, \alpha_m$ are scalars from the field $\mathbb{F}$. The *linear span* (or simply *span*) of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$, denoted by $\operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, is the set of all linear combinations of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$. In other words,

$$\operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \quad = \quad \Big\{ \sum_{i=1}^{m} \alpha_i \mathbf{v}_i \mid \alpha_1, \ldots, \alpha_m \in \mathbb{F} \Big\}.$$

As a special case, the empty sum of vectors is equal to the zero vector, and so $\operatorname{Span}(\emptyset) = \{\mathbf{0}\}$. Obviously, $\operatorname{Span}\{\mathbf{0}\} = \{\mathbf{0}\}$.

Suppose $\mathbf{v}$ is any non-zero vector in $\mathbb{R}^n$. Then $\operatorname{Span}\{\mathbf{v}\}$ is the **line** that passes through $\mathbf{v}$ (interpreted as a point) and the origin. (See the picture below for the case in $\mathbb{R}^2$.)



Next, suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are two non-zero vectors in $\mathbb{R}^n$. If neither of these two vectors is a scalar multiple of the other, then $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is the **plane** determined by the points $\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2$. On the other hand, if one of $\mathbf{v}_1, \mathbf{v}_2$ is a scalar multiple of the other, say $\mathbf{v}_2 = c\mathbf{v}_1$, then

$$\begin{aligned}
\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\} &= \operatorname{Span}\{\mathbf{v}_1, c\mathbf{v}_1\} \\
&= \{\alpha_1 \mathbf{v}_1, \alpha_2 (c\mathbf{v}_2) \mid \alpha_1, \alpha_2 \in \mathbb{R}\} \\
&= \{(\alpha_1 + c\alpha_2)\mathbf{v}_1 \mid \alpha_1, \alpha_2 \in \mathbb{R}\} \\
&\overset{(*)}{=} \{\alpha \mathbf{v}_1 \mid \alpha \in \mathbb{R}\} \\
&= \operatorname{Span}\{\mathbf{v}_1\},
\end{aligned}$$

where in (*), we used the fact that any scalar $\alpha \in \mathbb{R}$ can be represented in the form $\alpha_1 + c\alpha_2$, for some $\alpha_1, \alpha_2 \in \mathbb{R}$ (indeed, we can choose $\alpha_1 = \alpha$ and $\alpha_2 = 0$).

2 Matrix notation

Given a field $\mathbb{F}$, we denote by $\mathbb{F}^{n\times m}$ the set of all $n \times m$ matrices with entries in $\mathbb{F}$.[2] For example, the following is a matrix in $\mathbb{R}^{3\times 5}$:

$$\begin{bmatrix} -1 & 0 & -2 & 5 & 7 \\ 0 & 0 & 3/2 & -3 & 8 \\ \pi & 1 & \sqrt{3} & 0 & -5 \end{bmatrix}.$$

Schematically, there two common ways to represent a matrix A in $\mathbb{F}^{n\times m}$. First, we can represent it in the form

$$A \quad = \quad \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix},$$

where $\mathbf{a}_1, \dots, \mathbf{a}_m$ are column vectors in $\mathbb{F}^n$, and $\mathbf{a}_i$ is the i-th column of A (for $i = 1, \dots, m$).[3] Second, we can represent the matrix in the form

$$A \quad = \quad [a_{i,j}]_{n\times m}.$$

This notation indicates that the matrix is of size $n \times m$ (i.e. has n rows and m columns), and the i, j-th entry (i.e. the entry in the i-th row and j-th column) is $a_{i,j}$. So, if $A = [a_{i,j}]_{n\times m}$, then we have that

$$A \quad = \quad \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{bmatrix}.$$

3 Matrix-vector multiplication

Suppose that $\mathbb{F}$ is some field. Given a matrix $A \in \mathbb{F}^{n\times m}$ and a vector $\mathbf{x} \in \mathbb{F}^m$, say

$$A \quad = \quad \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix} \qquad \text{and} \qquad \mathbf{x} \quad = \quad \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix},$$

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[2] Reminder: An $n \times m$ *matrix* is a matrix with n rows and m columns.

[3] For example, if $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$, where $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\mathbf{a}_3 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, then $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 4 \end{bmatrix}$.

we define the matrix-vector product $A\mathbf{x}$ as follows:

$$A\mathbf{x} \quad := \quad \sum_{i=1}^{m} x_i \mathbf{a}_i \quad = \quad x_1\mathbf{a}_1 + \cdots + x_m\mathbf{a}_m.$$

Thus, $A\mathbf{x}$ is a linear combination of the columns of A, and the weights/scalars in front of the columns are determined by the entries of the vector $\mathbf{x}$.

Note that, for the matrix-vector product $A\mathbf{x}$ to be defined, two conditions must be satisfied:

- entries of the matrix A and entries of the vector $\mathbf{x}$ must belong to the same field;
- the number of **columns** of A must be the same as the number of entries of $\mathbf{x}$.

Schematically, we get

$$\underbrace{A}_{\in\mathbb{R}^{n\times m}} \; \underbrace{\mathbf{x}}_{\in\mathbb{R}^{m}} = \underbrace{A\mathbf{x}}_{\in\mathbb{R}^{n}}$$

Example 3.1. *Consider the matrix $A \in \mathbb{R}^{3\times 2}$ and vector $\mathbf{x} \in \mathbb{R}^2$, given below:*

$$A \quad = \quad \begin{bmatrix} -1 & 2 \\ 2 & 0 \\ 3 & -2 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{x} \quad = \quad \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Then

$$\begin{aligned} A\mathbf{x} \quad &= \quad \begin{bmatrix} -1 & 2 \\ 2 & 0 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ &= \quad 2\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \\ &= \quad \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}. \end{aligned}$$

Example 3.2. *Consider the matrix $A \in \mathbb{Z}_2^{2\times 3}$ and $\mathbf{x} \in \mathbb{Z}_2^3$, given below:*

$$A \quad = \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{x} \quad = \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Then

$$A\mathbf{x} \quad = \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad = \quad 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad = \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

4 Matrix-vector equations

A *matrix-vector equation* is an equation of the form $A\mathbf{x} = \mathbf{b}$, where the matrix $\mathbf{A}$ and vector $\mathbf{b}$ are known, and $\mathbf{x}$ is unknown. Once again, entries of A and $\mathbf{b}$ must come from the same field $\mathbb{F}$. Moreover, the number of **rows** of A must be the same as the number of entries of $\mathbf{b}$. Any solution $\mathbf{x}$ will then be a vector in $\mathbb{F}^n$, where n is the number of **columns** of A.

A matrix-vector equation is equivalent to a system of linear equations, as follows. Suppose that $\mathbb{F}$ is a field, $A \in \mathbb{F}^{n\times m}$ is a matrix, and $\mathbf{b} \in \mathbb{F}^n$ is a vector. Set $A = [a_{i,j}]_{n\times m}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$. We now transform the matrix-vector equation $A\mathbf{x} = \mathbf{b}$, as follows.

$$\begin{aligned}
& A\mathbf{x} = \mathbf{b} \\
\iff \quad & \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\
\iff \quad & x_1 \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{n,1} \end{bmatrix} + x_2 \begin{bmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{n,2} \end{bmatrix} + \cdots + x_m \begin{bmatrix} a_{1,m} \\ a_{2,m} \\ \vdots \\ a_{n,m} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\
\iff \quad & \begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,m}x_m \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,m}x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\
\iff \quad & \left\{ \begin{array}{ccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots + a_{1,m}x_m & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots + a_{2,m}x_m & = & b_2 \\ & & & & & \vdots & \\ a_{n,1}x_1 & + & a_{n,2}x_2 & + & \cdots + a_{n,m}x_m & = & b_n \end{array} \right.
\end{aligned}$$

So, solving a matrix-vector equation boils down to solving a system of linear equations. The augmented matrix of this linear system is the matrix

$$\left[\, A \;\vdots\; \mathbf{b} \,\right] = \left[\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \cdots & a_{1,m} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} & b_n \end{array} \right].$$

As in the case of linear systems, a matrix-vector equation $A\mathbf{x} = \mathbf{b}$ can have no solutions, one solution, or more than one solution. A matrix-vector equation that has at least one solution is *consistent*; a matrix-vector equation that has no solutions is *inconsistent*.

Example 4.1. *Solve the matrix-vector equation*

$$A\mathbf{x} \;=\; \mathbf{b},$$

where

$$A \;=\; \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \textit{and} \quad \mathbf{b} \;=\; \begin{bmatrix} 2 \\ 6 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$*.*

Solution. The augmented matrix of the linear system that corresponds to this matrix-vector equation is

$$\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right] \;=\; \left[\begin{array}{cc:c} 1 & 2 & 2 \\ 3 & 6 & 6 \end{array}\right].$$

We now row reduce in order to find $RREF(\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right])$.

$$\left[\begin{array}{cc:c} 1 & 2 & 2 \\ 3 & 6 & 6 \end{array}\right] \overset{R_3 \to R_3 - 3R_1}{\sim} \left[\begin{array}{cc:c} 1 & 2 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

The second matrix is in reduced row echelon form, and it corresponds to the following linear system.

$$\begin{array}{rcrcl} x_1 & + & 2x_2 & = & 2 \\ & & 0 & = & 0 \end{array}$$

The system is consistent, with one free variable (namely, x_2). We read off the solutions as follows.

$$\begin{array}{rcl} x_1 & = & -2s + 2 \\ x_2 & = & s, \qquad\qquad \text{where } s \in \mathbb{R}. \end{array}$$

So, the general solution of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} \;=\; \begin{bmatrix} -2s + 2 \\ s \end{bmatrix}, \qquad \text{where } s \in \mathbb{R}.$$

Here is another way to write the general solution of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$:

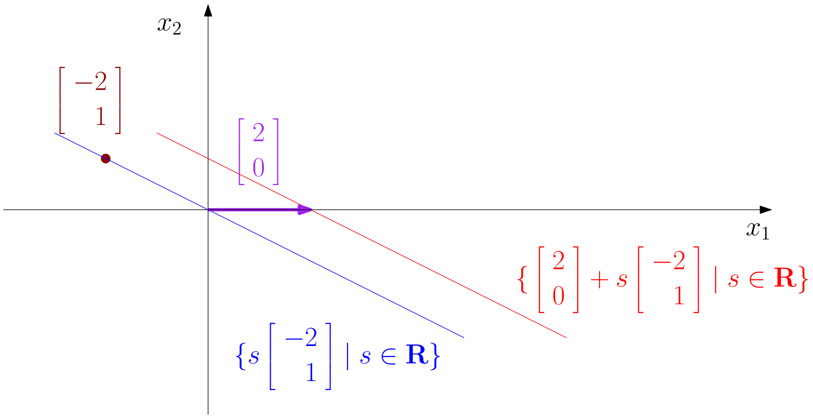
$$\mathbf{x} \;=\; \begin{bmatrix} 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad \text{where } s \in \mathbb{R}.$$

The set of solutions of the equation $A\mathbf{x} = \mathbf{b}$ is

$$\left\{ \begin{bmatrix} -2s + 2 \\ s \end{bmatrix} \mid s \in \mathbb{R} \right\} \;=\; \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \end{bmatrix} \mid s \in \mathbb{R} \right\}.$$

□

We remark that the solution set from Example 4.1 has a geometric interpretation. Indeed, $\left\{ s \begin{bmatrix} -2 \\ 1 \end{bmatrix} \mid s \in \mathbb{R} \right\} = \mathrm{Span}\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \}$ is the line that passes through the origin and the point $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ (this is the blue line in the picture below). The solution set $\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \end{bmatrix} \mid s \in \mathbb{R} \right\}$ is obtained by shifting this line by the vector $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, i.e. by adding the vector $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ to each point on the line (this vector is shown in purple in the picture below). The solution set $\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \end{bmatrix} \mid s \in \mathbb{R} \right\}$ is the red line in the picture below.



Example 4.2. *Solve the matrix-vector equation*

$$A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 \end{bmatrix} \quad \textit{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_3$*.*

Solution. The augmented matrix of the linear system that corresponds to this matrix-vector equation is

$$\left[\begin{array}{c:c} A & \mathbf{b} \end{array} \right] = \left[\begin{array}{cccc:c} 1 & 2 & 0 & 1 & 2 \\ 1 & 0 & 1 & 0 & 2 \\ 2 & 2 & 1 & 1 & 0 \end{array} \right].$$

We now row reduce in order to find $RREF(\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right])$.

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 2 \\ 1 & 0 & 1 & 0 & 2 \\ 2 & 2 & 1 & 1 & 0 \end{array}\right] \underset{\sim}{\overset{R_2\to R_2+2R_1}{\scriptstyle R_3\to R_3+R_1}} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 & 2 \end{array}\right]$$

$$\underset{\sim}{R_3\to R_3+2R_2} {\color{red}\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]}$$

$$\underset{\sim}{R_3\to 2R_3} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{R_1\to R_1+R_3} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{R_1\to R_1+R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The last matrix is in reduced row echelon form. Because the last column (the one after the dotted line) is a pivot column, we see that the corresponding system of linear equation has no solutions. Consequently, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has no solutions.

The solution set of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is $\emptyset$. ☐

Remark: In the solution of Example 4.2, we could in fact have stopped as soon as we got the red matrix (despite the fact that this matrix is not in reduced row echelon form). This is because the last row of the red equation encodes the equation $0 = 2$, which is impossible. Indeed, as soon as we obtain a row of the form $\left[\begin{array}{ccc|c} 0 & 0 & 0 & c \end{array}\right]$, where $c \neq 0$, we can stop row reducing, and we can deduce that the system has no solutions (because this row encodes the equation $0 = c$, which is a contradiction).

Example 4.3. *Solve the matrix-vector equation*

$$A\mathbf{x} \quad = \quad \mathbf{b},$$

where

$$A \quad = \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \textit{and} \quad \mathbf{b} \quad = \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_{\not\geq}$.

Solution. The augmented matrix of the linear system that corresponds to this matrix-vector equation is

$$\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right] \;=\; \left[\begin{array}{cc:c} 1 & 1 & 1 \\ 1 & 0 & 1 \end{array}\right].$$

We now row reduce in order to find $RREF(\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right])$.

$$\begin{aligned} \left[\begin{array}{cc:c} 1 & 1 & 1 \\ 1 & 0 & 1 \end{array}\right] & \overset{R_2 \to R_2 + R_1}{\sim} \left[\begin{array}{cc:c} 1 & 1 & 1 \\ 0 & 1 & 0 \end{array}\right] \\ & \overset{R_1 \to R_1 + R_2}{\sim} \left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right]. \end{aligned}$$

The last matrix is in reduced row echelon form. We now see that the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, namely

$$\mathbf{x} \;=\; \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The solution set of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. ☐

Example 4.4. *Solve the matrix-vector equation*

$$A\mathbf{x} \;=\; \mathbf{b},$$

where

$$A \;=\; \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \textit{and} \quad \mathbf{b} \;=\; \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

with entries understood to be in $\mathbb{Z}_2$.

Solution. The augmented matrix of the linear system that corresponds to this matrix-vector equation is

$$\left[\begin{array}{c:c} A & \mathbf{b} \end{array}\right] \;=\; \left[\begin{array}{cccc:c} 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right].$$

We now row reduce in order to find $RREF([\ A \mid \mathbf{b}\ ])$.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right] \overset{R_2 \to R_2 + R_1}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\overset{R_3 \to R_3 + R_2}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\overset{R_1 \to R_1 + R_2}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The final matrix is in reduced row echelon form, and it corresponds to the following linear system.

$$\begin{array}{ccccccccc} x_1 & & & + & x_3 & & & = & 0 \\ & & x_2 & + & x_3 & + & x_4 & = & 1 \\ & & & & & & 0 & = & 0 \end{array}$$

The system is consistent, with two free variables (namely, x_3 and x_4). We read off its solution as follows.

$$\begin{array}{rcl} x_1 & = & s \\ x_2 & = & s + t + 1 \\ x_3 & = & s \\ x_4 & = & t \qquad\qquad \text{where } s, t \in \mathbb{Z}_2. \end{array}$$

So, the general solution of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = \begin{bmatrix} s \\ s + t + 1 \\ s \\ t \end{bmatrix}, \qquad \text{where } s, t \in \mathbb{Z}_2.$$

We can also write the general solution of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$

as follows:[4]

$$\mathbf{x} \;=\; \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{where } s, t \in \mathbb{Z}_2.$$

The solution set of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is

$$\left\{ \begin{bmatrix} s \\ s+t+1 \\ s \\ t \end{bmatrix} \mid s, t \in \mathbb{Z}_2 \right\} \;=\; \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \mid s, t \in \mathbb{Z}_2 \right\}.$$

$\square$

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[4] We obtained this by separating the constant part from the parts associated with each parameter:

$$\mathbf{x} \;=\; \begin{bmatrix} s \\ s+t+1 \\ s \\ t \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \\ t \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

where $s, t \in \mathbb{Z}_2$.