Linear Algebra 1: Notes on mathematical induction

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1 Mathematical induction

Mathematical Induction is a proof technique that can be used to prove that a certain statement holds for all natural numbers n.

Let $P(n)$ be a statement about the number n. In order to prove that $P(n)$ holds for every $n \in \mathbb{N}$, it suffices to prove the following two statements:

- (Base case) $P(1)$ is true;
- (Induction step) for every $n \in \mathbb{N}$, if $\underbrace{P(n) \text{ is true}}_{\text{“induction hypothesis”}}$, then $P(n+1)$ is true.

Why does this work? Here's the intuition: We are trying to prove an infinite sequence of statements, namely,

$$P(1), P(2), P(3), P(4), \ldots$$

By the base case, $P(1)$ is true. By the induction step, since $P(1)$ is true, $P(2)$ is also true. Again by the induction step, since $P(2)$ is true, so is $P(3)$. Once again by the induction step, since $P(3)$ is true, so is $P(4)$. And so on! Thus, $P(n)$ is true for all n.

Example 1.1. *Prove that* $1+2+\cdots+n = \frac{n(n+1)}{2}$ *for all* $n \in \mathbb{N}$.

Solution. Let $P(n)$ be the statement that $1+2+\cdots+n = \frac{n(n+1)}{2}$. Thus:

- $P(1)$ is the statement that $1 = \frac{1\cdot(1+1)}{2}$;
- $P(2)$ is the statement that $1+2 = \frac{2\cdot(2+1)}{2}$;
- $P(3)$ is the statement that $1+2+3 = \frac{3\cdot(3+1)}{2}$;
- etc.

We need to prove that the statement $P(n)$ is true for all n.

Base case: $n = 1$. Obviously, $1 = \frac{1\cdot(1+1)}{2}$. Thus, $P(1)$ is true.

Induction step: Fix $n \in \mathbb{N}$, and assume that $P(n)$ is true. We must show that $P(n+1)$ is true.

The induction hypothesis states that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. Using this, we must prove that $1 + 2 + \cdots + n + (n+1) = \frac{(n+1)((n+1)+1)}{2}$. We compute:

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= (1 + 2 + \cdots + n) + (n+1) \\ &= \tfrac{n(n+1)}{2} + (n+1) && \text{by the induction hypothesis} \\ &= (n+1)(\tfrac{n}{2} + 1) \\ &= \tfrac{(n+1)((n+1)+1)}{2}. \end{aligned}$$

Thus, $P(n+1)$ is true. This completes the induction. ☐

Example 1.2. *Prove that $3n < 2^n$ for all integers $n \geq 4$.*

Proof. This is a slight variation on the theme. Since we are proving the statement for $n \geq 4$, our base case is $n = 4$ (rather than $n = 1$).

Base case: $n = 4$. Clearly, $3 \cdot 4 = 12 < 16 = 2^4$.

Induction step: Fix $n \in \mathbb{N}$ such that $n \geq 4$, and assume that $3n < 2^n$. We must show that $3(n+1) < 2^{n+1}$. We observe the following:

$$\begin{aligned} 3(n+1) &= 3n + 3 \\ &< 2^n + 3 && \text{by the induction hypothesis} \\ &< 2^n + 2^2 \\ &< 2^n + 2^n && \text{because } n > 2 \\ &= 2^{n+1} \end{aligned}$$

Thus, the statement is true for $n + 1$. This completes the induction. ☐

Example 1.3. *The* Fibonacci numbers *are defined as follows:*

- $F(1) = F(2) = 1$*;*
- $F(n+2) = F(n) + F(n+1)$ *for all* $n \in \mathbb{N}$.

Prove that $F(n) = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$.

Solution. The general term is defined in terms of the previous two terms. Thus, instead of one base case, we have two: $n = 1$ and $n = 2$. (Remark: If the general term were defined in terms of, say, the previous fifteen terms, then we would have fifteen base cases!)

Base case: For $n = 1$, we have:

$$\tfrac{(1+\sqrt{5})^1-(1-\sqrt{5})^1}{2^1\sqrt{5}} \quad = \quad \tfrac{2\sqrt{5}}{2\sqrt{5}} \quad = \quad 1 \quad = \quad F(1).$$

For $n = 2$, we have:

$$\tfrac{(1+\sqrt{5})^2-(1-\sqrt{5})^2}{2^2\sqrt{5}} \quad = \quad \tfrac{(1+2\sqrt{5}+5)-(1-2\sqrt{5}+5)}{4\sqrt{5}} \quad = \quad \tfrac{4\sqrt{5}}{4\sqrt{5}} \quad = \quad 1 \quad = \quad F(2).$$

Thus, the statement is true for $n = 1$ and $n = 2$.

Induction step: Fix $n \in \mathbb{N}$, and assume inductively that the statement is true for n and $n + 1$. We must show that it is true for $n + 2$.

By the induction hypothesis, $F(n) = \tfrac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{2^n\sqrt{5}}$ and $F(n + 1) = \tfrac{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}$. We must show that $F(n + 2) = \tfrac{(1+\sqrt{5})^{n+2}-(1-\sqrt{5})^{n+2}}{2^{n+2}\sqrt{5}}$. Let's compute:

$$F(n+2)$$

$$= \quad F(n) + F(n+1)$$

by the definition of Fibonacci numbers

$$= \quad \tfrac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{2^n\sqrt{5}} + \tfrac{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}$$

by the induction hypothesis

$$= \quad \tfrac{4(1+\sqrt{5})^n-4(1-\sqrt{5})^n}{2^{n+2}\sqrt{5}} + \tfrac{2(1+\sqrt{5})(1+\sqrt{5})^n-2(1-\sqrt{5})(1-\sqrt{5})^n}{2^{n+2}\sqrt{5}}$$

$$= \quad \tfrac{(6+2\sqrt{5})(1+\sqrt{5})^n-(6-2\sqrt{5})(1-\sqrt{5})^n}{2^{n+2}\sqrt{5}}$$

$$= \quad \tfrac{(1+\sqrt{5})^2(1+\sqrt{5})^n-(1-\sqrt{5})^2(1-\sqrt{5})^n}{2^{n+2}\sqrt{5}}$$

$$= \quad \tfrac{(1+\sqrt{5})^{n+2}-(1-\sqrt{5})^{n+2}}{2^{n+2}\sqrt{5}}.$$

This completes the induction. □

Example 1.4. *Prove that for all non-negative integers k, we have that*

- $8^{4k+1} \equiv 8$ *(mod* 10*);*
- $8^{4k+2} \equiv 4$ *(mod* 10*);*

- $8^{4k+3} \equiv 2$ *(mod* 10*);*
- $8^{4k+4} \equiv 6$ *(mod* 10*).*

Solution. We proceed by induction on k. Since we need to prove the statement for all non-negative integers k, our base case is $k = 0$.

Base case: For $k = 0$, we have:

- $8^{4\cdot 0+1} = 8 \equiv_{10} 8$;
- $8^{4\cdot 0+2} = 8^2 = 64 \equiv_{10} 4$;
- $8^{4\cdot 0+3} = 8^3 = 8 \cdot 8^2 \overset{(*)}{\equiv}_{10} 8 \cdot 4 = 32 \equiv_{10} 2$, where for (*), we used the fact that $8^2 \equiv_{10} 4$ (proven above);
- $8^{4\cdot 0+4} = 8^4 = 8 \cdot 8^3 \overset{(*)}{\equiv}_{10} 8 \cdot 2 = 16 \equiv_{10} 6$, where for (*) we used the fact that $8^3 \equiv_{10} 2$ (proven above).

Thus, the claim is true for $k = 0$.

Induction step: Fix a non-negative integer k, and assume inductively that the statement is true for k.[1] We must show that it is true for $k + 1$.[2] We saw in the base case that $8^4 \equiv 6 \pmod{10}$, and consequently, for each $\ell \in \{1, 2, 3, 4\}$, we have that

$$8^{4(k+1)+\ell} \quad = \quad 8^4 \cdot 8^{4k+\ell} \quad \equiv_{10} \quad 6 \cdot 8^{4k+\ell}$$

In the following calculations, (*) follows from the line right above this paragraph, and (**) follows from the induction hypothesis. We compute:

- $8^{4(k+1)+1} \overset{(*)}{\equiv}_{10} 6 \cdot 8^{4k+1} \overset{(**)}{\equiv}_{10} 6 \cdot 8 = 48 \equiv_{10} 8$;
- $8^{4(k+1)+2} \overset{(*)}{\equiv}_{10} 6 \cdot 8^{4k+2} \overset{(**)}{\equiv}_{10} 6 \cdot 4 = 24 \equiv_{10} 4$;

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[1] So, we are assuming that the following hold:

- $8^{4k+1} \equiv 8 \pmod{10}$;
- $8^{4k+2} \equiv 4 \pmod{10}$;
- $8^{4k+3} \equiv 2 \pmod{10}$;
- $8^{4k+4} \equiv 6 \pmod{10}$.

[2] So, we must prove the following:

- $8^{4(k+1)+1} \equiv 8 \pmod{10}$;
- $8^{4(k+1)+2} \equiv 4 \pmod{10}$;
- $8^{4(k+1)+3} \equiv 2 \pmod{10}$;
- $8^{4(k+1)+4} \equiv 6 \pmod{10}$.

- $8^{4(k+1)+3} \overset{(*)}{\equiv}_{10} 6 \cdot 8^{4k+3} \overset{(**)}{\equiv}_{10} 6 \cdot 2 = 12 \equiv_{10} 2$;
- $8^{4(k+1)+4} \overset{(*)}{\equiv}_{10} 6 \cdot 8^{4k+4} \overset{(**)}{\equiv}_{10} 6 \cdot 6 = 36 \equiv_{10} 6$.

This completes the induction. ☐

2 Strong induction

We now discuss a type of induction (sometimes called "strong induction") that lacks a base case. Again, let $P(n)$ be a statement about the number n. In order to prove that $P(n)$ holds for every $n \in \mathbb{N}$, it suffices to prove the following:

- for every $n \in \mathbb{N}$, if $P(1), \ldots, P(n-1)$ are all true,[3] then $P(n)$ is true.

Why does this make sense? Here's the intuition. Suppose that we have proven the statement above (i.e. that for every $n \in \mathbb{N}$, if $P(1), \ldots, P(n-1)$ are all true, then $P(n)$ is true). For $n = 1$, the hypothesis is vacuously true,[4] and so it follows that $P(1)$ is true. Since $P(1)$ is true, $P(2)$ is true. Now $P(1), P(2)$ are true; so $P(3)$ is true. Now $P(1), P(2), P(3)$ are true; so $P(4)$ is true. And so on!

Example 2.1. *Prove that every integer $n \geq 2$ can be written as a product of prime numbers.*[5]

Proof. Fix an integer $n \geq 2$, and assume inductively that each of $2, \ldots, n-1$ can be written as a product of primes.[6] We must show that n can be written as a product of primes.

Clearly, n is either prime or composite.

Suppose first that n is prime. Then, obviously, n can be written as a product of primes.[7]

Suppose now that n is composite. Then there exist integers n_1, n_2 such that $2 \leq n_1, n_2 \leq n-1$ and $n = n_1 n_2$. By the induction hypothesis, n_1 and n_2 can be written as products of primes. Set $n_1 = p_1 \cdot \cdots \cdot p_k$ and $n_2 = q_1 \cdot \cdots \cdot q_\ell$, where $p_1, \ldots, p_k, q_1, \ldots, q_\ell$ are prime. Then $n = n_1 n_2 = p_1 \cdot \cdots \cdot p_k \cdot q_1 \cdot \cdots \cdot q_\ell$. Thus, n is the product of primes. ☐

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[3] In other words: "if $P(i)$ is true for all $i \in \mathbb{N}$ such that $i \leq n-1$"

[4] Indeed, there are no positive integers $i \in \mathbb{N}$ such that $i \leq 1$.

[5] This statement is known at the Fundamental Theorem of Arithmetic.

[6] In other words, we are assuming that for all integers m such that $2 \leq m \leq n-1$, m can be written as a product of primes. Note that if $n = 2$, then we are in fact not assuming anything because there are no integers m satisfying $2 \leq m \leq \underbrace{n-1}_{=1}$.

[7] Namely, $n = \underbrace{n}_{\text{prime}}$.