Linear Algebra 1:
HW#9

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due Thursday, January 5, 2023, at 10 am (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Alternatively (if you don't feel like typing or scanning), you may submit a hard copy of your HW in lecture or tutorial **before** the deadline. Please do **not** e-mail your HW to me. Please write your **name** on the top of the first page of your HW.

Problem 1 (30 points)**.** *Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times m}$ and $B \in \mathbb{F}^{m\times p}$.*

(a) [20 points] Prove that Col(AB) is a subspace of Col(A), and that Row(AB) is a subspace of Row(B).

(b) [10 points] Using part (a), prove that $\text{rank}(AB) \leq \min\Big\{\text{rank}(A), \text{rank}(B)\Big\}$, *i.e. that* $\text{rank}(AB) \leq \text{rank}(A)$ *and* $\text{rank}(AB) \leq \text{rank}(B)$.

Problem 2 (30 points)**.** *In this problem, you may use the result of Problem 1 even if you did not prove it (or proved it incorrectly). Let $\mathbb{F}$ be a field.*

(a) [10 points] Prove that if $A \in \mathbb{F}^{n\times m}$ and $B \in \mathbb{F}^{m\times n}$ satisfy $AB = I_n$, then $n \leq m$.

(b) [20 points] Prove that if matrices $A, B \in \mathbb{F}^{n\times n}$ satisfy $AB = I_n$, then A and B are invertible and are each other's inverses.

> ***Hint:*** *What can you say about the rank of A and B? And what does that tell you about invertibility? Once you've proven that A and B are invertible, you still need to prove that they are each other's inverses; you are given that $AB = I_n$, but you still need to show that $BA = I_n$.*

Problem 3 (20 points)**.** *Give an example of matrices* $A \in \mathbb{R}^{2\times 3}$ *and* $B \in \mathbb{R}^{3\times 2}$ *such that* $AB = I_2$.

Problem 4 (20 points)**.** *Let* U *and* V *be vector spaces over a field* $\mathbb{F}$*, and assume that* U *is non-trivial (i.e. has at least one non-zero vector) and finite-dimensional. Let* $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ *be a linearly independent set in* U*, and let* $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$.[1] *Prove that there exists a linear transformation* $f : U \to V$ *such that* $f(\mathbf{u}_1) = \mathbf{v}_1, \ldots, f(\mathbf{u}_k) = \mathbf{v}_k$.

> ***Hint:*** *This follows pretty easily from a combination of* ***two*** *results proven in the Lecture Notes. Which ones?*
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> ***Remark:*** *The assumption that* U *is finite-dimensional is not strictly necessary, i.e. the statement of the problem remains true even if* U *is infinite-dimensional. However, this could not be proven without some pretty significant additional theory (Zorn's lemma!), which is why you are told to assume* U *is finite-dimensional.*

[1] Here, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are arbitrary vectors in V. They are not necessarily pairwise distinct.