Linear Algebra 1:
HW#8

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due Thursday, December 22, 2022, at 10 am (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Alternatively (if you don't feel like typing or scanning), you may submit a hard copy of your HW in lecture or tutorial **before** the deadline. Please do **not** e-mail your HW to me. Please write your **name** on the top of the first page of your HW.

Problem 1 (25 points)**.** *In this problem, you may use a calculator or computer software for any row reducing that you perform. Let*

$$A := \begin{bmatrix} 1 & -1 & 3 & 7 & 1 \\ 1 & 0 & -1 & -3 & 2 \\ 1 & -1 & 1 & 1 & -3 \\ 1 & -2 & 5 & 11 & -4 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$*.*

(a) Compute $\text{rank}(A)$*.*

(b) Find a basis of $\text{Col}(A)$*.*

(c) Find a basis of $\text{Row}(A)$*.*

(d) Determine whether $\mathbf{x} := \begin{bmatrix} 4 & -5 & -2 & 7 \end{bmatrix}^T$ *belongs to* $\text{Col}(A)$*, and if so, express* $\mathbf{x}$ *as a linear combination of the basis vectors for* $\text{Col}(A)$ *that you found in part (b).*

(e) Determine whether $\mathbf{y} := \begin{bmatrix} 1 & -1 & 5 & 5 & 5 \end{bmatrix}$ *belongs to* $\text{Row}(A)$*, and if so, express* $\mathbf{y}$ *as a linear combination of the basis vectors for* $\text{Row}(A)$ *that you found in part (c).*

Problem 2 (30 points)**.**

(a) *Either construct a matrix $A \in \mathbb{R}^{2\times 2}$ such that Col(A) = Nul(A), or prove that no such matrix exists.*

(b) *Either construct a matrix $B \in \mathbb{R}^{3\times 3}$ such that Col(B) = Nul(B), or prove that no such matrix exists.*

Definition. *Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$. We define $A^0 := I_n$, and for all positive integers m, we define $A^m := \underbrace{A \cdot \dots \cdot A}_{n}$.*[1]

Definition. *Let $\mathbb{F}$ be a field. A matrix $A \in \mathbb{F}^{n\times n}$ is* nilpotent *if there exists a positive integer m such that $A^m = O_{n\times n}$.*[2]

Problem 3 (25 points)**.** *Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$ be a nilpotent matrix. Let m be the smallest positive integer such that $A^m = O_{n\times n}$.*[3] *Let $\mathbf{v} \in \mathbb{F}^n$ be such that $A^{m-1}\mathbf{v} \neq \mathbf{0}$.*[4] *Prove that the vectors*

$$A^0\mathbf{v}, A\mathbf{v}, A^2\mathbf{v}, \dots, A^{m-1}\mathbf{v}$$

are linearly independent.

Problem 4 (20 points)**.** *Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$ be a nilpotent matrix. Prove that $A^n = O_{n\times n}$. (You may use the statement of Problem 3 even if you did not prove it.)*

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[1] Technically, this is a recursive definition. We define:

- $A^0 := I_n$, and
- $A^{m+1} := A^m A$ for all integers $m \geq 0$.

[2] As usual, $O_{n\times n}$ is the $n \times n$ matrix, all of whose entries are zero (the zero is from the field $\mathbb{F}$).

[3] So, $A^{m-1} \neq O_{n\times n}$. It is possible that $A = O_{n\times n}$; in this case, $m = 1$ and $A^{m-1} = A^0 = I_n \neq O_{n\times n}$.

[4] At least one such vector $\mathbf{v}$ exists. Indeed, since $A^{m-1} \neq O_{n\times n}$, we see that A^{m-1} has at least one non-zero column. Since $A^{m-1}\mathbf{e}_1, \dots, A^{m-1}\mathbf{e}_n$ (where $\mathbf{e}_1, \dots, \mathbf{e}_n$ are the standard basis vectors in $\mathbb{F}^n$) are precisely the columns of A^{m-1}, it follows that at least one of $A^{m-1}\mathbf{e}_1, \dots, A^{m-1}\mathbf{e}_n$ is non-zero. However, you may **not** assume that the vector $\mathbf{v}$ from the statement of the problem is equal to one of $\mathbf{e}_1, \dots, \mathbf{e}_n$.