Linear Algebra 1:
HW#7

Irena Penev
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due Thursday, December 15, 2022, at 10 am (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Alternatively (if you don't feel like typing or scanning), you may submit a hard copy of your HW in lecture or tutorial **before** the deadline. Please do **not** e-mail your HW to me. Please write your **name** on the top of the first page of your HW.

Problem 1 (30 points)**.** *Prove parts (b), (c), and (d) of Proposition 2.3 from Lecture Notes 6.*

> ***Hint:*** *Imitate the proof of Proposition 1.2 from Lecture Notes 6.*

Problem 2 (20 points)**.** *Let V be a vector space over a field $\mathbb{F}$, and let $A = \{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ be a linearly dependent set of vectors.*[1] *Prove that there exists some index $i \in \{1, \dots, k\}$ such that $\mathbf{a}_i$ is a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_{i-1}$.*[2]

> *Hint: You should slightly adapt one direction of the proof of Proposition 1.4(a) from Lecture Notes 7.*

Problem 3. *[20 points] Let V be a vector space over a field $\mathbb{F}$, and let U and W be subspaces of V. Prove that*

$$U + W \quad := \quad \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U,\ \mathbf{w} \in W\}$$

is a subspace of V.

[1] linearly dependent = not linearly independent

[2] It is possible that $i = 1$. Recall that the "empty sum" is equal to the zero vector.

Problem 4 (30 points)**.** *Let V be a finite-dimensional vector space over a field $\mathbb{F}$, and let U and W be subspaces of V. Prove that*

$$dim(U \cap W) + dim(U + W) \quad = \quad dim(U) + dim(W).$$

> ***Remark:*** *By Problem 4 from HW#6, $U \cap W$ is a subspace of V. By Problem 3 (above), $U + W$ is a subspace of V. Since V is finite-dimensional, so are $U \cap W$ and $U + W$ (by Theorem 1.12 from Lecture Notes 7).*
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> ***Hint:*** *Start with a basis of $U \cap W$ and then extend it in a suitable way, and show that what you obtain is a basis of $U + W$.*