Linear Algebra 1: HW#6

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due Thursday, December 8, 2022, at 10 am (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Alternatively (if you don't feel like typing or scanning), you may submit a hard copy of your HW in lecture or tutorial **before** the deadline. Please do **not** e-mail your HW to me. Please write your **name** on the top of the first page of your HW.

Definition. *Let n be a positive integer, and let π be a permutation in S_n. An* inversion *in π is an ordered pair (p, q) of integers in $\{1, \dots, n\}$ such that $p < q$ and $\pi(p) > \pi(q)$.*[1]

Problem 1 (30 points)**.** *Let n be a positive integer. Prove that for any permutation $\pi \in S_n$, we have that $sgn(\pi) = (-1)^r$, where r is the number of inversions (see the definition above) in π.*

> ***Hint:*** *Proceed by induction on the number of inversions. In the induction step, show that if a permutation $\pi \in S_n$ has $k+1$ inversions, then there exists a permutation π' and a transposition (ij) in S_n such that $\pi = (ij) \circ \pi'$ and π' has k inversions.*[2]

[1] For example, the permutation $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$ has four inversions, namely $(1,4), (2,3), (2,4), (3,4)$.

[2] For example, for $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$, we can take $\pi' = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, and we observe that $\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}}_{=\pi} = (34) \circ \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}}_{=\pi'}$, and that π has four inversions, whereas π' has three.

Problem 2 (30 points). *Let n be a positive integer. Let $\pi \in S_n$, and consider the disjoint cycle decomposition of π (with cycles of length one included). Let k_{even} be the number of even cycles in this decomposition.*[3] *Prove that $sgn(\pi) = (-1)^{k_{even}}$.*

Problem 3 (15 points). *Consider the permutation $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 3 & 2 \end{pmatrix}$ in S_5. In this problem, you are asked to compute $sgn(\pi)$ in three different ways, as follows.*

(a) List all the inversions of π, and using the result of Problem 1, compute $sgn(\pi)$.

(b) Find the disjoint cycle decomposition of π, and compute $sgn(\pi)$ using the result of Problem 2.

(c) Express π as a composition of transpositions,[4] *and compute $sgn(\pi)$ using Proposition 3.6 from Lecture Notes 5.*

Problem 4 (25 points). *Let V be a vector space over a field $\mathbb{F}$, and let U and W be subspaces of V. Prove that $U \cap W$ is a subspace of V.*

Hint: *Use Theorem 2.7 from Lecture Notes 6.*

[3] An *even cycle* is a cycle of even length, i.e. a cycle with an even number of elements.
[4] There is more than one right answer. However, try not to use too many transpositions.