Linear Algebra 1:
HW#4

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due Thursday, November 17, 2022, at 10 am (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Alternatively (if you don't feel like typing or scanning), you may submit a hard copy of your HW in lecture or tutorial **before** the deadline. Please do **not** e-mail your HW to me. Please write your **name** on the top of the first page of your HW.

Problem 1 (20 points)**.** *Let* $A \in \mathbb{R}^{n\times m}$ *and* $\mathbf{b} \in \mathbb{R}^n$*. Assume that the matrix-vector equation* $A\mathbf{x} = \mathbf{b}$ *is consistent, and let* $\mathbf{x}_0$ *be some solution of the equation* $A\mathbf{x} = \mathbf{b}$*. Prove that the solution set of the equation* $A\mathbf{x} = \mathbf{b}$ *is precisely the set* $\{\mathbf{y} + \mathbf{x}_0 \mid \mathbf{y} \in \mathbb{R}^m, A\mathbf{y} = \mathbf{0}\}$*.*[1]

> ***Remark #1:*** *Actually, this is true for general fields* $\mathbb{F}$*, not just for* $\mathbb{R}$ *(and in fact, the proof is the same for general fields* $\mathbb{F}$ *as for* $\mathbb{R}$*). But, for the sake of simplicity, you are only asked to prove this for* $\mathbb{R}$*.*
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> ***Remark #2/Hint:*** *Essentially, you are asked to prove that, if* $A\mathbf{x} = \mathbf{b}$ *is consistent, and* $\mathbf{x}_0$ *is an arbitrarily chosen solution of* $A\mathbf{x} = \mathbf{b}$*, then the solution set of* $A\mathbf{x} = \mathbf{b}$ *is obtained by "shifting" the solution set of the homogeneous equation* $A\mathbf{x} = \mathbf{0}$ *by* $\mathbf{x}_0$*. Note that you must prove two things:*
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> 1. *if* $\mathbf{y} \in \mathbb{R}^m$ *is a solution of the homogeneous equation* $A\mathbf{x} = \mathbf{0}$*, then* $\mathbf{y} + \mathbf{x}_0$ *is a solution of* $A\mathbf{x} = \mathbf{b}$*;*
> 2. *any solution of* $A\mathbf{x} = \mathbf{b}$ *can be expressed in the form* $\mathbf{y} + \mathbf{x}_0$*, for some solution* $\mathbf{y}$ *of the homogeneous equation* $A\mathbf{x} = \mathbf{0}$*.*

[1] Here, $\mathbf{0}$ is the zero vector in $\mathbb{R}^n$.

Problem 2 (20 points). *Solve the linear system below, where k is some fixed constant. (The coefficients are assumed to be in $\mathbb{R}$.)*

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & 3x_3 & + & 2x_4 & = & -4 \\ 2x_1 & + & 5x_2 & & & + & 5x_4 & = & -3 \\ 2x_1 & + & 4x_2 & + & (k^2-5k)x_3 & + & (k+2)x_4 & = & k-11 \end{array}$$

> ***Remark:*** *Your solutions will depend on k. In fact, the **number** of solutions will also depend on k. You will need to figure out for which (if any) k the system has no solutions, for which (if any) it has a unique solution, and for which (if any) it has infinitely many solutions. If the system is consistent, make sure you specify the solutions!*

Problem 3 (20 points). *Let*

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \\ 1 & 4 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 3 & 2 \\ 8 & 6 & 4 \\ 12 & 8 & 6 \\ 17 & 12 & 9 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Solve the equation $AX = B$. How many solutions does the equation $AX = B$ have?

Problem 4 (20 points). *Let*

$$A = \begin{bmatrix} 2 & 2 & 2 & 0 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 1 \end{bmatrix}$$

with entries understood to be in $\mathbb{Z}_3$. Solve the equation $XA = B$. How many solutions does the equation $XA = B$ have?

Problem 5 (20 points). *Write the (invertible) matrix*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

(with entries understood to be in $\mathbb{R}$) as a product of elementary matrices.

> ***Remark:*** *There is more than one solution. Try not to use too many elementary matrices in your product, though.*