Linear Algebra 1: HW#3

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due Thursday, November 3, 2022, at 10 am (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Alternatively (if you don't feel like typing or scanning), you may submit a hard copy of your HW in lecture or tutorial **before** the deadline. Please do **not** e-mail your HW to me. Please write your **name** on the top of the first page of your HW.

Problem 1 (30 points)**.** *Let $\mathbb{F}$ be a field,*[1] *and let $f : \mathbb{F}^m \to \mathbb{F}^n$ be a function. Prove that the following are equivalent:*

(1) f is linear;

(2) for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{F}^m$ and all scalars $\alpha, \beta \in \mathbb{F}$, we have that

$$f(\alpha\mathbf{u} + \beta\mathbf{v}) \quad = \quad \alpha f(\mathbf{u}) + \beta f(\mathbf{v}).$$

Problem 2 (20 points)**.** *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Using the* ***definition*** *of a linear transformation, prove that for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$, we have that $f(\mathbf{u} - \mathbf{v}) = f(\mathbf{u}) - f(\mathbf{v})$.*[2]

> ***Remark:*** *Actually, this works for general fields $\mathbb{F}$, not just for $\mathbb{R}$. For the sake of simplicity, you are only asked to prove it for $\mathbb{R}$.*

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[1] For now, you may assume that $\mathbb{F}$ is either $\mathbb{R}$, or $\mathbb{C}$, or $\mathbb{Z}_p$ (for some prime p).

[2] Vector subtraction is defined the usual way, i.e. entry-by-entry. For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$, with $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$, we define $\mathbf{u} - \mathbf{v} = \begin{bmatrix} u_1 - v_1 \\ \vdots \\ u_m - v_m \end{bmatrix}$.

Problem 3 (30 points)**.** *Let $\mathbb{F}$ be a field,*[3] *let $f, g : \mathbb{F}^m \to \mathbb{F}^n$ be linear transformations, let $A, B \in \mathbb{F}^{n \times m}$ be the standard matrices of f, g, respectively, and let $c \in \mathbb{F}$ be a scalar.*

(a) The function $f+g : \mathbb{F}^m \to \mathbb{F}^n$ is defined by setting $(f+g)(\mathbf{u}) = f(\mathbf{u})+g(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{F}^m$. Prove that $f + g$ is linear, and find its standard matrix.

(b) The function $cf : \mathbb{F}^m \to \mathbb{F}^n$ is defined by setting $(cf)(\mathbf{u}) = c(f(\mathbf{u}))$ for all $\mathbf{u} \in \mathbb{F}^m$. Prove that cf is linear, and find its standard matrix.

Problem 4 (20 points)**.** *Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by*

$$f\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 + u_2 \\ u_2 + u_3 \end{bmatrix}$$

for all $u_1, u_2, u_3 \in \mathbb{R}$. Using either the ***definition*** *of a linear transformation or the result of Problem 1, prove that f is linear. Then, find the standard matrix of f.*

[3] For now, you may assume that $\mathbb{F}$ is either $\mathbb{R}$, or $\mathbb{C}$, or $\mathbb{Z}_p$ (for some prime p).