

# Linear Algebra 1: HW#1

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due Thursday, October 20, 2022, at 10 am (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Alternatively (if you don't feel like typing or scanning), you may submit a hard copy of your HW in lecture or tutorial **before** the deadline. Please do **not** e-mail your HW to me.

**Remark:** All the problems below should be solved using **induction** (either mathematical induction or strong induction, whichever is more convenient). **Solutions that do not use induction will receive no credit.**

**Problem 1** (25 points). *Prove that the number  $2^{n+1} + 3^{2n-1}$  is divisible by 7 for all positive integers  $n$ .*

**Problem 2** (25 points). *Prove that for all real numbers  $a_1, \dots, a_n$  such that  $0 \leq a_1, \dots, a_n < 1$ , we have that*

$$(1 - a_1)(1 - a_2) \dots (1 - a_n) \geq 1 - a_1 - a_2 - \dots - a_n.$$

**Problem 3** (25 points). *Show that any collection of at least 5 cities can be connected via one-way airlines<sup>1</sup> in such a way that any city is reachable from any other city with at most one layover.*

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<sup>1</sup>So, for any two cities,  $A$  and  $B$ , at most one of the following (direct) flights is possible:

1. from  $A$  to  $B$ ;
2. from  $B$  to  $A$ .

**Problem 4** (25 points). *Prove that every positive integer has a binary representation. More precisely, prove that for any positive integer  $n$ , there exists a sequence  $a_0, \dots, a_k$  of 0's and 1's, with  $a_k = 1$ , such that*

$$n = \sum_{i=0}^k a_i \cdot 2^i$$

*(so,  $\overline{a_k a_{k-1} \dots a_0}$  is the binary representation of  $n$ ). You may use the fact that for any positive integer  $n$ , there exists a non-negative integer  $\ell$  such that  $2^\ell \leq n < 2^{\ell+1}$ .*