NDMI012: Combinatorics and Graph Theory 2
Tutorial 11

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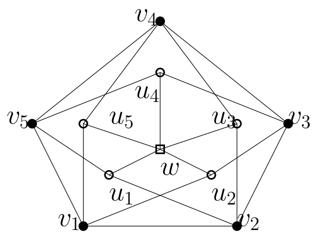
Exercise 1. *Let G be a graph that has a Hamiltonian path. Prove that for any set $S \subsetneq V(G)$, the graph $G \setminus S$ has at most $|S| + 1$ many components.*

Exercise 2. *Determine if the Grötzch graph (below) is Hamiltonian, and if so, find a Hamiltonian cycle in it.*

> ***Remark:*** *The Grötzch graph is the same as the Mycielski graph M_4.*



The Grötzch graph

Exercise 3. *Prove that the Chvátal closure of a graph is uniquely defined.*

Definition. *The* Cartesian product *of graphs G and H, denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H)$ and edge set*

$$\{(u, v_1)(u, v_2) \mid u \in V(G), v_1 v_2 \in E(H)\} \cup \{(u_1, v)(u_2, v) \mid u_1 u_2 \in E(G), v \in E(H)\}.$$

Exercise 4. *Prove that the Cartesian product of two Hamiltonian graphs is again Hamiltonian.*

Exercise 5. *Let G be a Hamiltonian bipartite graph, and let $x, y \in V(G)$. Prove that $G \setminus \{x, y\}$ has a perfect matching if and only if x and y are on the opposite sides of the bipartition of G. Apply this to prove that deleting two unit squares from an 8×8 chessboard leaves a board that can be partitioned into 1×2 rectangles if and only if the two missing squares have opposite colors.*

Definition. *The* girth *of a graph that is not a forest is the length of its shortest cycle.*

Exercise 6. *Let G be a graph that is not a forest, and assume that G has girth at least five. Prove that $\overline{G}$ is Hamiltonian.*

> ***Hint:*** *Use Ore's condition.*