NDMI012: Combinatorics and Graph Theory 2
Tutorial 10

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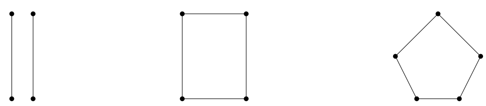
Definition. *A* star-cutset *of a graph G is a set $S \subsetneqq V(G)$ such that both the following hold:*

- *there is a vertex $s \in S$ such that $S \subseteq N_G[s]$;*[1]
- *$G \setminus S$ is disconnected.*

Exercise 3(b) from Tutorial 9. *Let G be a graph, all of whose proper induced subgraphs are perfect. Assume that G admits a star-cutset. Prove that G is perfect.*

Definition. *A graph is* split *if its vertex set can be partitioned into a clique and a stable set.*[2]

Exercise 1. *Prove that a graph is split if and only if it is $\{2K_2, C_4, C_5\}$-free.*[3]



$2K_2$ C_4 C_5

> ***Hint for the "if" ("$\Longleftarrow$") direction:*** *Assume that a graph G is $\{2K_2, C_4, C_5\}$-free. If G is not a complete graph, then of all maximum cliques of G, choose K to be one for which the number of edges in $G \setminus K$ is as small as possible. Now show that $V(G) \setminus K$ is a stable set.*

[1] In other words, some vertex of S is adjacent to all other vertices of S.

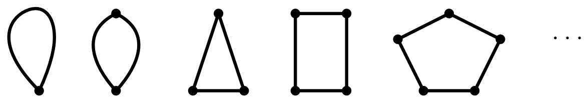
[2] The clique or the stable set may possibly be empty.

[3] For a family of graphs $\mathcal{H}$, a graph G is *$\mathcal{H}$-free* if no induced subgrpah of G is isomorphic to a graph in $\mathcal{H}$.

Exercise 2. *Prove that a graph G is split if and only if G and $\overline{G}$ are both chordal.*

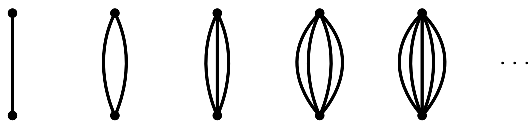
Exercise 3.

(a) *[10 points] Prove that for all $n \geq 1$, $T_{C_n}(x, y) = y + \sum_{i=1}^{n-1} x^i$.*



C_1 C_2 C_3 C_4 C_5 $\dots$

(b) *[10 points] Prove that for $n \geq 1$, $T_{I_n}(x, y) = x + \sum_{i=1}^{n-1} y^i$, where I_n is the 2-vertex multigraph with n edges between them.*



I_1 I_2 I_3 I_4 I_5 $\dots$

Hint: *Induction on n, using the recursive deletion-contraction formula for the Tutte polynomial.*