NDMI012: Combinatorics and Graph Theory 2
Tutorial 9

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Exercise 1. *Let G be a graph. Prove that the following are equivalent:*

(1) G is perfect;

(2) every induced subgraph H of G contains a stable set that intersects all maximum cliques of H.

Definition. *A graph G is* strongly perfect *if every induced subgraph H of G has a stable set that intersects all maximal cliques of H.*

Exercise 2. *By Exercise 1, every strongly perfect graph is perfect. Find an example of a perfect graph that is **not** strongly perfect.*

Definition. *A* star-cutset *of a graph G is a set $S \subsetneqq V(G)$ such that both the following hold:*

- *there is a vertex $s \in S$ such that $S \subseteq N_G[s]$;*[1]
- *$G \setminus S$ is disconnected.*

Exercise 3. *Let G be a graph, all of whose proper induced subgraphs are perfect.*

(a) Assume that G admits a clique-cutset. Prove that G is perfect.

(b) Assume that G admits a star-cutset. Prove that G is perfect.

Definition. *Given graphs G and H, we say that G is* H-free *if no induced subgraph of G is isomorphic to H.*

[1] In other words, some vertex of S is adjacent to all other vertices of S.

Exercise 4. *As usual, P_4 is the path on four vertices and three edges. Prove that if G is a P_4-free graph on at least two vertices, then G or $\overline{G}$ is disconnected.*[2]

Proof. **Hint:** Induction on the number of vertices. □

Exercise 5. *Using Exercise 4 (or in some other way), prove that every P_4-free graph is strongly perfect.*

> ***Hint:*** *Induction on the number of vertices.*

[2] $\overline{G}$ = complement of G