NDMI012: Combinatorics and Graph Theory 2
Tutorial 8

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Exercise 1. *Let n be a positive integer, let $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$ be such that $a_1 < b_1, \dots, a_n < b_n$, and let $I_1 = (a_1, b_1), \dots, I_n = (a_n, b_n)$. Let G be the graph with vertex set $\{v_1, \dots, v_n\}$, and with adjacency as follows: for all distinct $i, j \in \{1, \dots, n\}$, v_i is adjacent to v_j in G if and only if $I_i \cap I_j \neq \emptyset$.*[1] *Prove that G is a chordal graph.*

> ***Hint:*** *Induction on n. In the induction step, find a simplicial vertex, delete it, and apply the induction hypothesis.*

Exercise 2. *Let T be a tree, and let $T_1, \dots, T_k$, with $k \geq 1$, be (not necessarily distinct) subtrees of T. Assume that for all distinct $i, j \in \{1, \dots, k\}$, $V(T_i) \cap V(T_j) \neq \emptyset$. Prove that $V(T_1) \cap \dots \cap V(T_k) \neq \emptyset$.*

> ***Hint:*** *Induction on $|V(T)|$.*

Exercise 3. *Let T be a tree, and let $T_1, \dots, T_k$, with $k \geq 1$, be (not necessarily distinct) subtrees of T. Let G be the graph with vertex-set $V(G) = \{v_1, \dots, v_k\}$, and adjacency as follows: for all distinct $i, j \in \{1, \dots, k\}$, $v_i v_j \in E(G)$ if and only if $V(T_i) \cap V(T_j) \neq \emptyset$. Prove that G is chordal.*

> ***Hint:*** *Induction on $|V(T)|$.*

Exercise 4. *Let G be a chordal graph, and set $V(G) = \{v_1, \dots, v_n\}$. Prove that there exists a tree T and (not necessarily distinct) subtrees $T_1, \dots, T_n$ of T such that for all distinct $i, j \in \{1, \dots, n\}$, $v_i v_j \in E(G)$ if and only if $V(T_i) \cap V(T_j) \neq \emptyset$.*

> ***Hint:*** *Induction on n.*

Exercise 5. *Do Exercise 4 with the additional requirement that $\Delta(T) \leq 3$, and that $T_1, \dots, T_n$ are pairwise distinct.*

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[1] A graph obtained in this way is called an *interval graph.*