NDMI012: Combinatorics and Graph Theory 2
Tutorial 7

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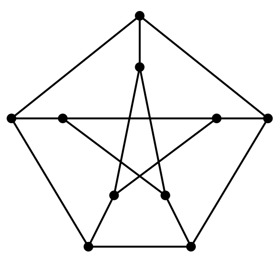
Exercise 1. *Compute the chromatic index of the Petersen graph.*



Petersen graph

Exercise 2. *Edge coloring (and proper edge coloring) can be defined for loopless multigraphs in the natural way.*[1] *Prove that every loopless multigraph G satisfies $\chi'(G) \leq 2\Delta(G)$. Do **not** use Shannon's theorem (stated below).*

Shannon's theorem. *Every loopless multigraph G satisfies $\chi'(G) \leq \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor$.*

Proof. Omitted. □

Exercise 3. *For each integer Δ, construct a multigraph G of maximum degree Δ, and satisfying $\chi'(G) = \lfloor \frac{3\Delta}{2} \rfloor$.*

> ***Hint:*** *It's probably best to first do this for even Δ. Then slightly adapt your construction to get odd Δ.*
>
> ***Remark:*** *This proves that the bound for Shannon's theorem is optimal.*

[1] So, we consider graphs that have no loops, but may have parallel edges.

Exercise 4. *Prove that for any triangle-free graph H, there exists some integer $k \geq 2$ such that H is isomorphic to an induced subgraph of the Mycielski graph M_k.*

> ***Hint:*** *Induction on $|V(H)|$.*

Definition. *We define the sequence $\{Z_k\}_{k=1}^{\infty}$ of* Zykov graphs *recursively, as follows. Let $Z_1 := K_1$. Next, fix a positive integer k, and assume inductively that graphs $Z_1, \dots, Z_k$ have been defined. We define Z_{k+1} as follows. We first take the disjoint union of $Z_1, \dots, Z_k$, and then we add $\prod_{i=1}^{k} |V(Z_i)|$ "new" vertices, each corresponding to a unique combination of selections of one vertex from each of $Z_1, \dots, Z_k$.*[2] *Make each new vertex adjacent to the k vertices it corresponds to (and to no other vertices). The resulting graph is Z_{k+1}.*

Exercise 5. *Prove that for all positive integers k, the Zykov graph Z_k is triangle-free and has chromatic number k.*

[2] There are precisely $\prod_{i=1}^{k} |V(Z_i)|$ such selections, and for each selection, we have exactly one "new" vertex that corresponds to it.