NDMI012: Combinatorics and Graph Theory 2
Tutorial 6

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Exercise 6 from Tutorial 3. *Let G be a 3-connected graph on at least six vertices, and assume that G contains K_5 as a topological minor. Prove that G contains $K_{3,3}$ as a topological minor.*

Exercise 1. *Prove that for every graph G, there exists an ordering $v_1, \ldots, v_n$ of its vertices such that the greedy algorithm applied to G with the ordering $v_1, \ldots, v_n$ produces an optimal coloring of G, i.e. a proper coloring that uses only $\chi(G)$ colors.*

Exercise 2. *For every integer $n \geq 2$, construct a bipartite graph on $2n$ vertices, ordered in such a way that the greedy algorithm uses n colors.*

Definition. *A graph G is* critical *if all its proper induced subgraphs H satisfy $\chi(H) < \chi(G)$.*[1]

Exercise 3. *Determine all critical graphs of chromatic number 3.*

Exercise 4. *Prove that every graph G satisfies*

$$\chi(G) \quad \leq \quad \max\{\delta(H) + 1 \mid H \text{ is a subgraph of } G\}.$$

Exercise 5. *Let G be a graph whose odd cycles are pairwise intersecting, i.e. every two odd cycles of G share at least one vertex. Prove that $\chi(G) \leq 5$.*

[1] In some texts, a graph G is defined to be critical if all its proper subgraphs H (not necessarily induced) satisfy $\chi(H) < \chi(G)$. Here, we will use the definition with induced subgraphs.