NDMI012: Combinatorics and Graph Theory 2
Tutorial 5

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Definition. *A graph is* maximally planar *if it is planar, and it is not a proper subgraph of any planar graph on the same vertex set.*[1]

Definition. *A* minimal non-planar graph *is a graph that is not planar, but all of its proper subgraphs are planar.*

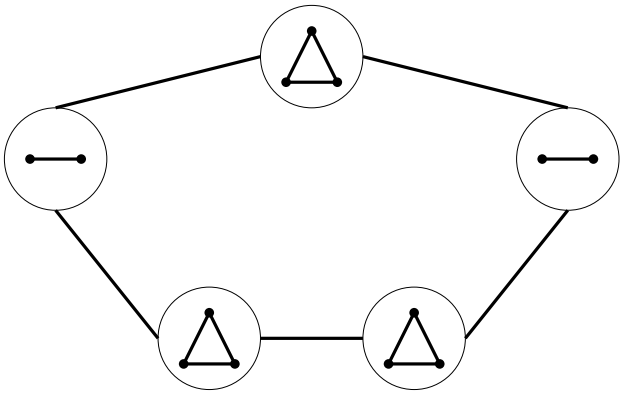
Exercise 5 from Tutorial 3. *Does every minimal non-planar graph G contain an edge e such that $G - e$ is maximally planar? Does the answer change if we define "minimal" with respect to minors rather than subgraphs?*

Exercise 6 from Tutorial 3. *Let G be a 3-connected graph on at least six vertices, and assume that G contains K_5 as a topological minor. Prove that G contains $K_{3,3}$ as a topological minor.*

Exercise 1 from Tutorial 4. *Verify that the graph below is a counterexample to Hajós' Conjecture for $k = 7$,*[2] *that is, show that the graph has chromatic number 7, but does not contain K_7 as a topological minor. Then, show that this graph is* ***not*** *a counterexample to Hadwiger's Conjecture, that is, show that it contains K_7 as a minor.*

[1] This means that the graph is planar, but turning any non-edge of the graph into an edge produces a non-planar graph.

[2] A line between two circles indicates that all vertices inside one of the circles are adjacent to all vertices inside the other circle.



Exercise 1. *Draw the following graphs on the following surfaces (without edge crossings):*

- $K_{3,3}$ *on the torus;*
- $K_{3,3}$ *on the projective plane;*
- K_6 *on the Klein bottle.*