NDMI012: Combinatorics and Graph Theory 2

Lecture #13

Extremal combinatorics

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- the Sunflower lemma.

Part I: Turán's theorem

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Definition

Given a positive integer n and a graph H, an n-vertex graph G without an H subgraph is *extremal* (for the property of not containing H as a subgraph) if it has the largest possible number of edges among all n-vertex graphs without an H subgraph; $\mathrm{ex}(n, H)$ is the number of edges of an extremal n-vertex graph without an H subgraph.

- So, $\mathrm{ex}(n, H)$ is the maximum number of edges that an n-vertex graph that does not contain H as a subgraph can have.

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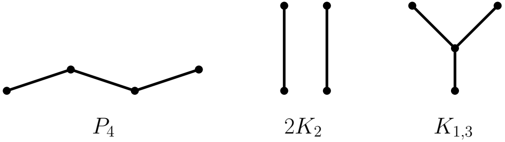
- The converse, however, does not hold in general: it is possible that a graph is edge-maximal without an H subgraph, without being extremal.
  - For example, $2K_2$ is a four-vertex edge-maximal graph without a P_4 subgraph, but it is not extremal: indeed, $K_{1,3}$ also has four vertices and no P_4 subgraph, and it has more edges than $2K_2$.



Mantel's theorem

For any positive integer n, we have that $\mathrm{ex}(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$, and moreover, $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is an extremal n-vertex graph without a K_3 subgraph.

Proof. Combinatorics & Graphs 1.

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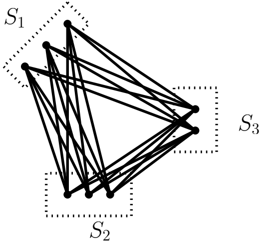
Proof. Combinatorics & Graphs 1.

- Mantel's theorem is a special case of Turán's theorem, to which we now turn.

Definition

For a positive integer r, a *complete r-partite graph* is a graph G whose vertex set can be partitioned into r (possibly empty) stable sets (called *parts*), pairwise complete to each other.

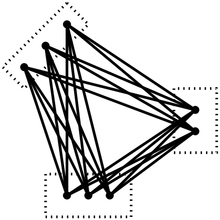


Definition

A *complete multipartite graph* is any graph that is complete r-partite for some r.

Definition

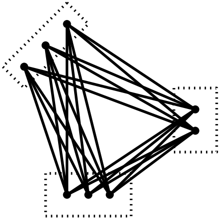
The *r-partite Turán graph on n vertices*, denoted by $T_r(n)$, is the complete r-partite graph on n vertices, in which the sizes of any two parts differ by at most one (so, each part is of size $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$); $t_r(n)$ is the number of edges of $T_r(n)$.



$T_3(8)$

Turán's theorem

Let n and r be positive integers. Then $ex(n, K_{r+1}) = t_r(n)$, and furthermore, $T_r(n)$ is the unique (up to isomorphism) extremal n-vertex graph without a K_{r+1} subgraph.



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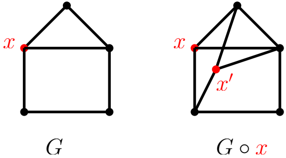
- *Duplicating a vertex* x of a graph G produces a supergraph $G \circ x$ by adding to G a vertex x' and making it adjacent to all the neighbors of x in G, and to no other vertices of G (in particular, x and x' are nonadjacent in $G \circ x$).



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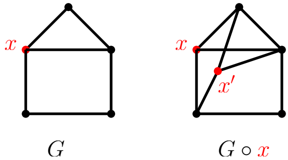
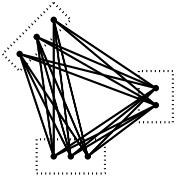


- Obviously, $\omega(G \circ x) = \omega(G)$, i.e. G contains K_{r+1} is a subgraph iff $G \circ x$ does.



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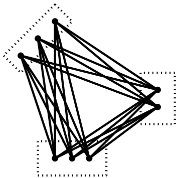
Proof.



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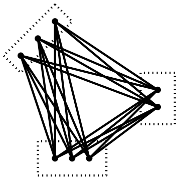
Proof. We may assume that $r < n$, for otherwise, $T_r(n) \cong K_n$, and the result is immediate.



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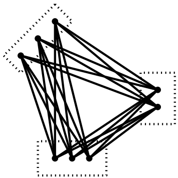

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It is clear that $T_r(n)$ is an n-vertex graph without a K_{r+1} subgraph. Now, let G be any n-vertex extremal graph without a K_{r+1} subgraph. We must show that $G \cong T_r(n)$.

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Proof (continued). Reminder: $r < n$; G is n-vertex extremal graph without a K_{r+1} subgraph.

Claim. *G is a complete multipartite graph.*

Proof of the Claim.

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So, $d_G(y_1) \leq d_G(x)$, and similarly, $d_G(y_2) \leq d_G(x)$.

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$|E(G')| = |E(G)| - (d_G(y_1) + d_G(y_2)) + 1 + 2d_G(x) \geq |E(G)| + 1$,
contrary to the fact that G is extremal. This proves the Claim.

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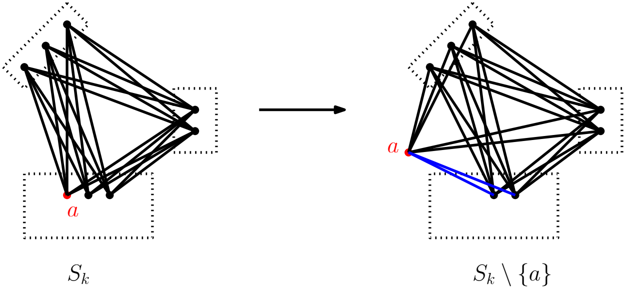

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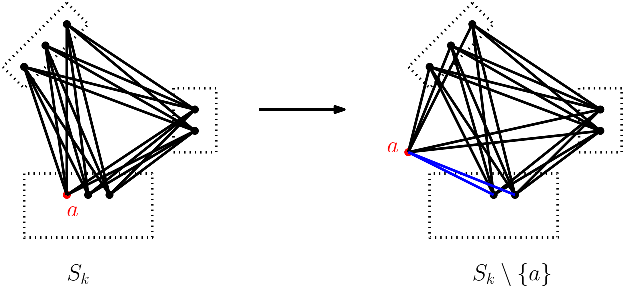

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So, $G \cong T_r(n)$.

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  - In that case, we can fix any $x \in \{1, \ldots, n\}$, and consider all r-element subsets of $\{1, \ldots, n\}$ that contain x.
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  - As the following theorem shows, this is in fact best possible.

Part II: The Erdős-Ko-Rado theorem

- Suppose we are given positive integers r and n, and we want to select a maximum number of pairwise intersecting r-element subsets of $\{1, \ldots, n\}$.
  - What is this maximum number?
- For $r > \frac{n}{2}$, any two r-element subsets of $\{1, \ldots, n\}$ intersect, and there are $\binom{n}{r}$ many such subsets.
- How about if $r \leq \frac{n}{2}$?
  - In that case, we can fix any $x \in \{1, \ldots, n\}$, and consider all r-element subsets of $\{1, \ldots, n\}$ that contain x.
  - There are $\binom{n-1}{r-1}$ many such subsets, and obviously, they pairwise intersect.
  - As the following theorem shows, this is in fact best possible.

The Erdős-Ko-Rado theorem

Let r and n be positive integers s.t. $r \leq \frac{n}{2}$. Then there are at most $\binom{n-1}{r-1}$ many pairwise distinct and pairwise intersecting r-element subsets of $\{1, \ldots, n\}$.

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Proof. Let $A_1, \ldots, A_m$ be pairwise distinct and pairwise intersecting r-element subsets of $\{1, \ldots, n\}$. WTS $m \leq \binom{n-1}{r-1}$.

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Let c be the number of ordered pairs (C, A), where

- C is a directed cycle with vertex set $\{1, \dots, n\}$;
  - vertics $1, \dots, n$ need **not** appear in that order on the cycle;
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Now we count in two ways, as follows. On the one hand, we can form an ordered pair (C, A) by first selecting one of the sets $A_1, \dots, A_m$ (we have m choices), then ordering its vertices to form a directed path (there are $r!$ choices), and then ordering the remaining $n - r$ vertices to complete the cycle C (there are $(n - r)!$ choices). So, $c = mr!(n - r)!$.

Proof (continued). Reminder: $A_1, \ldots, A_m$ are pairwise distinct and pairwise intersecting r-element subsets of $\{1, \ldots, n\}$; c is the number of ordered pairs (C, A), where

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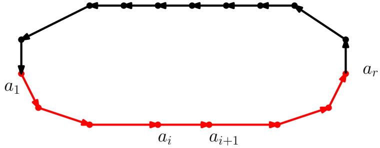
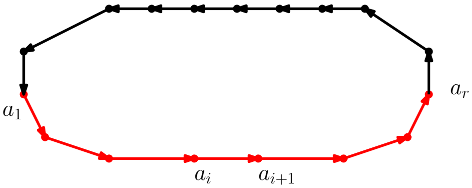
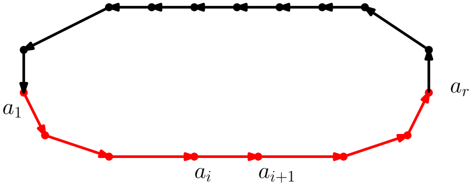
We now count in another way. First, there are $(n-1)!$ ways of ordering $\{1, \dots, n\}$ to obtain a directed cycle C. WTS that for fixed C, there are at most r directed subpaths of C that correspond to one of A_i's. Indeed, suppose the subpath $a_1, a_2, \dots, a_r$ of C corresponds to one of $A_1, \dots, A_m$.





Then for any other subpath of C corresponding to one of $A_1, \ldots, A_m$ (and therefore containing at least one of $a_1, \ldots, a_r$), there exists some $i \in \{1, \ldots, r-1\}$ s.t. either a_i is the terminal vertex of the path, or a_{i+1} is the initial vertex of the path;



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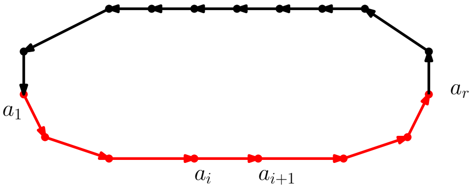

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$$c \leq (n-1)!r.$$

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Let r and n be positive integers s.t. $r \leq \frac{n}{2}$. Then there are at most $\binom{n-1}{r-1}$ many pairwise distinct and pairwise intersecting r-element subsets of $\{1, \ldots, n\}$.

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So,

$$m \quad \leq \quad \frac{(n-1)!r}{r!(n-r)!} \quad = \quad \binom{n-1}{r-1},$$

which is what we needed to show.

Part III: The Sunflower lemma

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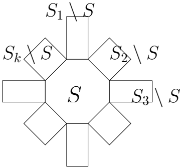
Definition

A *sunflower* is a family (i.e. collection) $\mathscr{S}$ of sets (called *petals*) s.t. there exists a set S (called a *kernel*) with the property that for all distinct $S_1, S_2 \in \mathscr{S}$, we have that $S_1 \cap S_2 = S$.

- A sunflower $\mathscr{S} = \{S_1, \dots, S_k\}$ with kernel S:



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The Sunflower lemma [Erdős-Rado]

Let ℓ and p be positive integers, and let $\mathscr{A}$ be a family of sets s.t.

- $|A| \leq \ell$ for all $A \in \mathscr{A}$, and
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Then there exists a sunflower $\mathscr{S} \subseteq \mathscr{A}$ with p petals.

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Proof. We assume inductively that the lemma is true for smaller values of ℓ. If $p \leq 2$ or $\ell = 1$, then it's easy (details: Lecture Notes). So, we assume that $p \geq 3$ and $\ell \geq 2$.

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Claim. *There exists some $d \in D$ s.t. d belongs to more than $(p-1)^{\ell-1}(\ell-1)!$ many elements of $\mathscr{A}$.*

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Then every element of $\mathscr{A}$ intersects D: indeed, since $\emptyset \notin \mathscr{D}$, we know that every element of $\mathscr{D}$ intersects D, and by the maximality of $\mathscr{D}$, every element of $\mathscr{A} \setminus \mathscr{D}$ intersects D.

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$$\left\lceil \frac{|\mathscr{A}|}{|D|} \right\rceil \quad > \quad \frac{(p-1)^{\ell}\ell!}{(p-1)\ell} \quad = \quad (p-1)^{\ell-1}(\ell-1)!$$

many elements of $\mathscr{A}$, which is what we needed. This proves the Claim.

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Let $d \in D$ be as in the Claim, and set $\mathscr{A}' := \{A \setminus \{d\} \mid A \in \mathscr{A}, d \in A\}$.

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- $|\mathscr{A}| > (p-1)^\ell \ell!$.

Then there exists a sunflower $\mathscr{S} \subseteq \mathscr{A}$ with p petals.

Proof (continued). Reminder: $p \geq 3$, $\ell \geq 2$; true for smaller values of ℓ; $\mathscr{D} \subseteq \mathscr{A}$ be a collection of pairwise disjoint sets of maximum possible size; $D = \bigcup \mathscr{D}$.

> ***Claim.*** *There exists some $d \in D$ s.t. d belongs to more than $(p-1)^{\ell-1}(\ell-1)!$ many elements of $\mathscr{A}$.*

Let $d \in D$ be as in the Claim, and set $\mathscr{A}' := \{A \setminus \{d\} \mid A \in \mathscr{A}, d \in A\}$. Then $|\mathscr{A}'| > (p-1)^{\ell-1}(\ell-1)!$, and $|A| \leq \ell - 1 \; \forall A \in \mathscr{A}'$. So, by the ind. hyp., $\exists$ sunflower $\mathscr{S}' \subseteq \mathscr{A}'$ with p petals. Now, set $\mathscr{S} := \{A \cup \{d\} \mid A \in \mathscr{S}'\}$; then $\mathscr{S} \subseteq \mathscr{A}$ is a sunflower with p petals, and we are done.