NDMI012: Combinatorics and Graph Theory 2

Lecture #12

Pólya enumeration theorem. Exponential generating functions

Irena Penev

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This lecture consists of two parts:

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1. the Pólya enumeration theorem;

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1. the Pólya enumeration theorem;
2. an introduction to exponential generating functions.

Part I: Pólya enumeration

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Part I: Pólya enumeration

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Definition

A *subgroup* of a group G is a subset of G that is a group under the operation inherited from G.

- Every group is a subgroup of itself, as is the one-element group consisting only of the identity element.

Definition

Let X be a set of size n, and let G be a subgroup of $\mathrm{Sym}(X)$. Each element of G can be represented as a composition of disjoint cycles, the sum of whose lengths is n. Now, for $g \in G$ and $k \in \{1, \ldots, n\}$, we denote by $j_k(g)$ the number of cycles of length k, when g is written as a composition of disjoint cycles.[a] For $g \in G$, we set $x^{\mathrm{cs}(g)} := x_1^{j_1(g)} x_2^{j_2(g)} \ldots x_n^{j_n(g)}$. Finally, the *cycle index* of the group G is

$$\mathcal{Z}_G(x_1, \ldots, x_n) \;=\; \frac{1}{|G|} \sum_{g \in G} x^{\mathrm{cs}(g)}.$$

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[a] For example, if $n = 7$ and $g = (124)(35)(6)(7)$, then $j_1(g) = 2$, $j_2(g) = 1$, $j_3(g) = 1$, and $j_4(g) = j_5(g) = j_6(g) = j_7(g) = 0$. Do not forget to count cycles of length one!

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Example 1.1

Compute cycle index of the group Sym(2).

Solution.

- Cycle index: $\mathcal{Z}_G(x_1, \ldots, x_n) = \frac{1}{|G|} \sum_{g \in G} x^{\mathrm{cs}(g)}$.

Example 1.1

Compute cycle index of the group Sym(2).

Solution. Here, using the notation from the definition of a cycle index, we have that $X = \{1, 2\}$ and $n = 2$. We have that $\mathrm{Sym}(2) = \{(1)(2), (12)\}$, and clearly,

- $x^{\mathrm{cs}\big((1)(2)\big)} = x_1^2 x_2^0 = x_1^2$;
- $x^{\mathrm{cs}\big((12)\big)} = x_1^0 x_2^1 = x_2$.

So,

$$\mathcal{Z}_{\mathrm{Sym}(2)}(x_1, x_2) = \frac{x_1^2 + x_2}{2}.$$

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Example 1.2

Compute cycle index of the group Sym(3).

Solution.

- Cycle index: $\mathcal{Z}_G(x_1, \ldots, x_n) = \frac{1}{|G|} \sum_{g \in G} x^{\mathrm{cs}(g)}$.

Example 1.2

Compute cycle index of the group Sym(3).

Solution. Here, using the notation from the definition of a cycle index, we have that $X = \{1, 2, 3\}$ and $n = 3$. Sym(3) has one element that is a composition of three 1-cycles; it has three elements that are a composition of one 2-cycle and one 1-cycle; and it has two elements that consist of one 3-cycle. So,

$$\mathcal{Z}_{\mathrm{Sym}(3)}(x_1, x_2, x_3) = \frac{x_1^3 + 3x_1x_2 + 2x_3}{6}.$$

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$$\pi'(\{i,j\}) \;=\; \{\pi(i), \pi(j)\},$$

and we set $\mathrm{Sym}'(n) = \{\pi' \mid \pi \in \mathrm{Sym}(n)\}$.

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  and we set $\mathrm{Sym}'(n) = \{\pi' \mid \pi \in \mathrm{Sym}(n)\}$.
- It is easy to check that $\mathrm{Sym}'(n)$ is a subgroup of $\mathrm{Sym}\Big(\binom{\{1,\ldots,n\}}{2}\Big)$.
- In particular, every permutation in $\mathrm{Sym}'(n)$ can be represented as a composition of disjoint cycles, the sum of whose lengths is $\binom{n}{2}$.

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Example 1.3

Compute cycle index of the group $\mathrm{Sym}'(5)$.

Solution. We remark that $\binom{5}{2} = 10$, and so each permutation in $\mathrm{Sym}'(5)$ can be represented as a composition of disjoint cycles, the sum of whose lengths is 10.

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We analyze the cycle structure of permutations in $\mathrm{Sym}(5)$: given the cycle structure of a permutation $\pi \in \mathrm{Sym}(5)$, we describe the cycle structure of π'.

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Compute cycle index of the group $\mathrm{Sym}'(5)$.

Solution (continued).

- There is one permutation π in $\mathrm{Sym}(5)$ (namely, the identity permutation) of the form $(a)(b)(c)(d)(e)$. For such a π, we have that π' is the composition of ten cycles of length one. So, $x^{\mathrm{cs}(\pi')} = x_1^{10}$.

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- There are 10 permutations π in $\mathrm{Sym}(5)$ of the form $(ab)(c)(d)(e)$. For such a π, we see that π' has three cycles of the length two (these cycles are of the form $(\{a, x\}, \{b, x\})$, with $x \notin \{a, b\}$), and it has four cycles of length one. So, $x^{\mathrm{cs}(\pi')} = x_1^4 x_2^3$.

- Cycle index: $\mathcal{Z}_G(x_1, \ldots, x_n) = \frac{1}{|G|} \sum_{g \in G} x^{\mathrm{cs}(g)}$.

Example 1.3

Compute cycle index of the group $\mathrm{Sym}'(5)$.

Solution (continued).

- There are 15 permutation π in $\mathrm{Sym}(5)$ of the form $(ab)(cd)(e)$. For such a π, we see that π' has exactly two cycles of length one (namely, $(\{a, b\})$ and $(\{c, d\})$), and the remaining cycles of π' (four of them) are of length two. So, $x^{\mathrm{cs}(\pi')} = x_1^2 x_2^4$.

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- There are 20 permutations π in $\mathrm{Sym}(5)$ of the form $(abc)(d)(e)$. For such a π, we see that π' has one cycle of length one (namely, $(\{d,e\})$), and the remaining cycles of π' (three of them) are of length three. So, $x^{\mathrm{cs}(\pi')} = x_1 x_3^3$.

- Cycle index: $\mathcal{Z}_G(x_1, \ldots, x_n) = \frac{1}{|G|} \sum_{g \in G} x^{\mathrm{cs}(g)}$.

Example 1.3

Compute cycle index of the group $\mathrm{Sym}'(5)$.

Solution (continued).

- There are 20 permutations π in $\mathrm{Sym}(5)$ of the form $(abc)(de)$. For such a π, we see that π' has one cycle of length one (namely, $(\{d, e\})$), one cycle of length three (namely, $(\{a, b\}, \{b, c\}, \{c, a\})$), and one cycle of length six (containing all the remaining elements of $\binom{\{1,\ldots,5\}}{2}$). So, $x^{\mathrm{cs}(\pi')} = x_1 x_3 x_6$.

- Cycle index: $\mathcal{Z}_G(x_1, \ldots, x_n) = \frac{1}{|G|} \sum_{g \in G} x^{\mathrm{cs}(g)}$.

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Compute cycle index of the group $\mathrm{Sym}'(5)$.

Solution (continued).

- There are 30 permutations π in $\mathrm{Sym}(5)$ of the form $(abcd)(e)$. For such a π, we see that π' has two 4-cycles (namely, $(\{a,e\},\{b,e\},\{c,e\},\{d,e\})$ and $(\{a,b\},\{b,c\},\{c,d\},\{d,a\})$) and one 2-cycle (namely, $(\{a,c\},\{b,d\})$). So, $x^{\mathrm{cs}(\pi')} = x_2 x_4^2$.

- Cycle index: $\mathcal{Z}_G(x_1, \ldots, x_n) = \frac{1}{|G|} \sum_{g \in G} x^{\mathrm{cs}(g)}$.

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- There are 24 permutations π in $\mathrm{Sym}(5)$ of the form $(abcde)$. For such a π, we see that π' has two 5-cycles (namely, $(\{a,b\},\{b,c\},\{c,d\},\{d,e\},\{e,a\})$ and $(\{a,c\},\{b,d\},\{c,e\},\{d,a\},\{e,b\})$). So, $x^{\mathrm{cs}(\pi')} = x_5^2$.

- Cycle index: $\mathcal{Z}_G(x_1, \ldots, x_n) = \frac{1}{|G|} \sum_{g \in G} x^{\mathrm{cs}(g)}$.

Example 1.3

Compute cycle index of the group $\mathrm{Sym}'(5)$.

Solution (continued). Since $|\mathrm{Sym}'(5)| = |\mathrm{Sym}(5)| = 5! = 120$, we now see that

$$\mathcal{Z}_{\mathrm{Sym}'(5)}(x_1, \ldots, x_{10})$$

$$= \frac{1}{120}\Big(x_1^{10} + 10x_1^4x_2^3 + 15x_1^2x_2^4 + 20x_1x_3^3 + 20x_1x_3x_6 + + 30x_2x_4^2 + 24x_5^2\Big).$$

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- Let $\mathcal{C}$ be the set of all colorings of X using the color set C (formally, $\mathcal{C}$ is simply the set of all functions from X to C).
- Then G acts on $\mathcal{C}$ in the natural way: for all $\pi \in G$, $c \in C$, and $x \in X$, we set $(\pi \cdot c)(x) = c(\pi^{-1} \cdot x)$; the idea is that $\pi \cdot c$ should assign to x the color that c assigned to the element of X that got "moved" to x via π, i.e. to the element $\pi^{-1} \cdot x$.

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- Two colorings are *equivalent* if one can be transformed into the other via our group action, i.e. if they belong to the same orbit of our action.
- Now, let $\mathcal{D} \subseteq \mathcal{C}$.

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  - The *coloring inventory* of $\mathcal{D}$ is a polynomial in $c_1, \dots, c_k$, which is the sum of terms of the form $c_1^{d_1} \dots c_k^{d_k}$, and the coefficient in front of the term $c_1^{d_1} \dots c_k^{d_k}$ is the number of colorings in $\mathcal{D}$ that, for each $i \in \{1, \dots, k\}$, assign color c_i to precisely d_i elements of X.

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  - The *pattern inventory* of $\mathcal{D}$ is a polynomial in $c_1, \ldots, c_k$, which is the sum of terms of the form $c_1^{d_1} \ldots c_k^{d_k}$, and the coefficient in front of the term $c_1^{d_1} \ldots c_k^{d_k}$ is the number of **non-equivalent** colorings in $\mathcal{D}$ that, for each $i \in \{1, \ldots, k\}$, assign color c_i to precisely d_i elements of X.

- Cycle index: $\mathcal{Z}_G(x_1, \ldots, x_n) = \frac{1}{|G|} \sum_{g \in G} x^{\mathrm{cs}(g)}$.

Pólya enumeration theorem

Let $C = \{c_1, \ldots, c_k\}$ be a set of colors, let X be a finite set of size n, and let G be a subgroup of $\mathrm{Sym}(X)$, acting on X in the natural way.[a] Let $\mathcal{C}$ be the set of all colorings of X with colors from C, and let G act on $\mathcal{C}$ in the natural way.[b] Then the pattern inventory of $\mathcal{C}$ is $\mathcal{Z}_G(\sum_{i=1}^k c_i, \sum_{i=1}^k c_i^2, \ldots, \sum_{i=1}^k c_i^n)$.

[a] This means that for all $\pi \in \mathrm{Sym}(X)$ and $x \in X$, we have that $\pi \cdot x = \pi(x)$.

[b] That is, for all $\pi \in G$, $c \in C$, and $x \in X$, we set $(\pi \cdot c)(x) = c(\pi^{-1} \cdot x)$.

- Proof: Lecture Notes (uses Burnside's lemma).

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- Proof: Lecture Notes (uses Burnside's lemma).
- Let's look at some examples.

- Cycle index: $\mathcal{Z}_G(x_1, \ldots, x_n) = \frac{1}{|G|} \sum_{g \in G} x^{\mathrm{cs}(g)}$.
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Compute the number of non-equivalent colorings of a bracelet with four beads, using colors black and white for the beads. (Two colorings are equivalent if one can be transformed into the other via a rotation or a reflection.)

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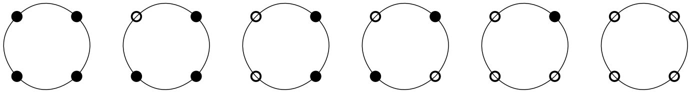
- In this particular case, it is easy to see that there are exactly six non-equivalent colorings, represented below.



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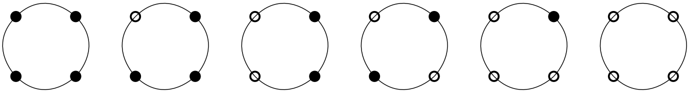

- However, let us apply the Pólya enumeration theorem in order to illustrate the principle.

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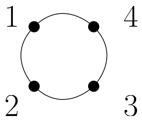
Solution. We label the beads $1, 2, 3, 4$ counterclockwise.



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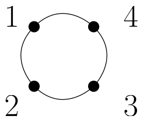

The group acting on the beads is simply the dihedral group D_8 (symmetries of the square).

- Cycle index: $\mathcal{Z}_G(x_1, \ldots, x_n) = \frac{1}{|G|} \sum_{g \in G} x^{\mathrm{cs}(g)}$.
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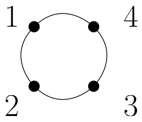
Example 1.5

Compute the number of non-equivalent colorings of a bracelet with four beads, using colors black and white for the beads. (Two colorings are equivalent if one can be transformed into the other via a rotation or a reflection.)

Solution. We label the beads $1, 2, 3, 4$ counterclockwise.



The group acting on the beads is simply the dihedral group D_8 (symmetries of the square). The elements of the group are (next slide):

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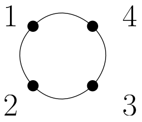
Solution (continued). The elements of the group are:



- (1)(2)(3)(4)
- (1234)
- (13)(24)
- (1432)
- (12)(34)
- (14)(23)
- (1)(24)(3)
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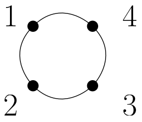

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So, $\mathcal{Z}_{D_8}(x_1, \ldots, x_4) = \frac{1}{8}(x_1^4 + 2x_1^2 x_2 + 3x_2^2 + 2x_4)$.

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Solution (continued). Reminder:
$\mathcal{Z}_{D_8}(x_1, \ldots, x_4) = \frac{1}{8}(x_1^4 + 2x_1^2 x_2 + 3x_2^2 + 2x_4)$.

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$\mathcal{Z}_{D_8}(x_1, \ldots, x_4) = \frac{1}{8}(x_1^4 + 2x_1^2 x_2 + 3x_2^2 + 2x_4)$.
Consequently,

$$\begin{aligned}
& \mathcal{Z}_{D_8}(b + w, b^2 + w^2, b^3 + w^3, b^4 + w^4) \\
= \; & \frac{1}{8}\Big((b+w)^4 + 2(b+w)^2(b^2+w^2) + 3(b^2+w^2)^2 + 2(b^4+w^4)\Big) \\
= \; & b^4 + b^3 w + 2b^2 w^2 + b w^3 + w^4.
\end{aligned}$$

- Cycle index: $\mathcal{Z}_G(x_1, \ldots, x_n) = \frac{1}{|G|} \sum_{g \in G} x^{\mathrm{cs}(g)}$.
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The total number of colorings is equal to the sum of coefficients of the polynomial above: $1+1+2+1+1=6$.

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- **Remark:** The polynomial above allows us to do more, namely, to count the number of non-equivalent colorings with a fixed number of black and white beads (details: Lecture Notes).

- For each positive integer n and permutation $\pi \in \mathrm{Sym}(n)$, we define a permutation π' on the set $\binom{\{1,\ldots,n\}}{2}$ by setting$\pi'(\{i,j\}) = \{\pi(i), \pi(j)\}$, and we set $\mathrm{Sym}'(n) = \{\pi' \mid \pi \in \mathrm{Sym}(n)\}$.

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Proposition 1.6

Let $n \geq 2$ and $k \geq 0$ be integers. Then the number of non-isomorphic graphs on n vertices and k edges is equal to the coefficient in front of the term x^k in the polynomial $\mathcal{Z}_{\mathrm{Sym}'(n)}(1+x, 1+x^2, \ldots, 1+x^{\binom{n}{2}})$.

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Proof. Let $\mathcal{C}$ be the set of all colorings of the set $\binom{\{1,\dots,n\}}{2}$ using the color set $\{b, w\}$. We let $\mathrm{Sym}'(n)$ act on $\mathcal{C}$ in the natural way. Now, colorings in $\mathcal{C}$ correspond to n-vertex graphs in the natural way: the vertex-set is $\{1, \dots, n\}$, and edges are pairs colored b ("black"), where as the non-edges are the pairs colored w ("white").

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Proof (continued). The number of non-isomorphic five-vertex graphs with k edges is precisely the number of non-equivalent colorings in $\mathcal{C}$ (with respect to our group action) in which exactly k elements of $\binom{\{1,\ldots,n\}}{2}$ are colored b (and the remaining $\binom{n}{2} - k$ elements are colored w).

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Proof (continued). By the Pólya enumeration theorem, the latter is precisely the coefficient in front of $b^k w^{\binom{n}{2}-k}$ in the polynomial $\mathcal{Z}_{\mathrm{Sym}'(5)}(b+w, b^2+w^2, \ldots, b^{\binom{n}{2}} + w^{\binom{n}{2}})$.

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$$\mathcal{Z}_{\mathrm{Sym}'(n)}(1 + x, 1 + x^2, \ldots, 1 + x^{\binom{n}{2}})$$

(we replace b by x and w by 1).

Example 1.7

For each non-negative integer k, find the number of non-isomorphic k-edge graphs on five vertices.

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Solution. We apply Proposition 1.6. By Example 1.3, we know that

$$\begin{aligned} & \mathcal{Z}_{\mathrm{Sym}'(5)}(x_1, \dots, x_{10}) \\ = \; & \tfrac{1}{120}\Big(x_1^{10} + 10x_1^4x_2^3 + 15x_1^2x_2^4 + 20x_1x_3^3 + 20x_1x_3x_6 + \\ & + 30x_2x_4^2 + 24x_5^2\Big), \end{aligned}$$

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\end{aligned}$$

and so

$$\begin{aligned}
&\mathcal{Z}_{\mathrm{Sym}'(5)}(1+x, \ldots, 1+x^{10}) \\
= \;& 1 + x + 2x^2 + 4x^3 + 6x^4 + 6x^5 + 6x^6 + 4x^7 + 2x^8 + x^9 + x^{10}.
\end{aligned}$$

Example 1.7

For each non-negative integer k, find the number of non-isomorphic k-edge graphs on five vertices.

Solution. Thus, up to isomorphism,

- there is one edgeless graph on five vertices;
- there is one graph on five vertices with one edge;
- there are two graphs on five vertices with two edges;
- there are four graphs on five vertices with three edges;
- there are six graphs on five vertices with four edges;
- there are six graphs on five vertices with five edges;
- there are six graphs on five vertices with six edges;
- there are four graphs on five vertices with seven edges;
- there are two graphs on five vertices with eight edges;
- there is one graph on five vertices with nine edges;
- there is one graph on five vertices with ten edges;
- there are no graphs on five vertices with more than ten edges.

Part II: Exponential generating functions

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- Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real (or complex) numbers.

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- The *exponential generating function* (abbreviated *egf*) of $\{a_n\}_{n=0}^{\infty}$ is the function

$$g(x) \;=\; \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \;=\; \frac{a_0}{0!} + \frac{a_1 x}{1!} + \frac{a_2 x^2}{2!} + \frac{a_3 x^3}{3!} + \ldots$$

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- Here, we give a brief introduction to exponential generating functions.
- We begin with a simple example, in which we contrast the use of ogf’s and egf’s.

Example 2.1

(a) Find the number of ways that three letters from the word SEQUENCE can be selected (order does not matter).[a]

(b) Find the number of ways that three letters from the word SEQUENCE can be arranged (order matters).[b]

---

[a] Note that the letter E appears three times, and so we may select between zero and three copies of E. The three E's are considered the same: so, if we select (say) two E's, we do not care which particular two we selected.

[b] For example, SEE and ESE count as different. However, the E's are still interchangeable: we do not care which of the three E's from the word SEQUENCE correspond to the two E's from SEE.

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Solution. The number of ways we can select three letters from the word SEQUENCE is the coefficient in front of x^3 in the polynomial

$$f(x) \quad = \quad (1 + x + x^2 + x^3)(1 + x)^5,$$

which is 26. (Here, the polynomial $1 + x + x^2 + x^3$ corresponds to the letter E, and the five terms $1 + x$ correspond to the remaining five letters of the word SEQUENCE.)

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More generally, the coefficient in front of x^k in $f(x)$ is the number of ways we can select k letters from the word SEQUENCE (when order does not matter).

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More generally, the coefficient in front of x^k in $f(x)$ is the number of ways we can select k letters from the word SEQUENCE (when order does not matter). So in fact, $f(x)$ is the ogf for the sequence $\{a_k\}_{k=0}^{\infty}$, where a_k is the number of ways of selecting k letters from the word SEQUENCE (when order does not matter).

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(b) Find the number of ways that three letters from the word SEQUENCE can be arranged (order matters).

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Solution. Here, we use an egf. The number of ways we can arrange three letters from the word SEQUENCE is the coefficient in front of $\frac{x^3}{3!}$ in the polynomial

$$g(x) = (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!})(1 + x)^5,$$

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Indeed, for each $k \in \{0, 1, 2, 3\}$, we select k E's and $3 - k$ of the remaining five letters. The number of ways of selecting those $3 - k$ other letters is precisely the coefficient in front of x^{3-k} in $(1+x)^5$, and then the number of ways of arranging our three chosen letters (k E's and $3 - k$ other letters) is $\frac{3!}{k!}$.

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Indeed, for each $k \in \{0, 1, 2, 3\}$, we select k E's and $3 - k$ of the remaining five letters. The number of ways of selecting those $3 - k$ other letters is precisely the coefficient in front of x^{3-k} in $(1 + x)^5$, and then the number of ways of arranging our three chosen letters (k E's and $3 - k$ other letters) is $\frac{3!}{k!}$. So, the total number of ways of arranging three letters from the word SEQUENCE is precisely the coefficient in front of $\frac{x^3}{3!}$ in $g(x)$.

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Solution (continued). Reminder: $g(x) = (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!})(1 + x)^5$.

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More generally, the coefficient in front of $\frac{x^k}{k!}$ in $g(x)$ is the number of ways we can arrange k letters from the word SEQUENCE (when order matters). So in fact, $g(x)$ is the egf for the sequence $\{b_k\}_{k=0}^{\infty}$, where b_k is the number of ways of arranging k letters from the word SEQUENCE (when order matters).

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Find the ogf and egf of the constant sequence $1, 1, 1, 1, \ldots$.

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Solution. The ogf of the sequence is

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

whereas the egf of the sequence is

$$g(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

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  which has radius of convergence 0, i.e. the series only converges for $x = 0$.
- On the other hand, the egf of the sequence is

$$g(x) \;=\; \sum_{n=0}^{\infty} \frac{n! x^n}{n!} \;=\; \sum_{n=0}^{\infty} x^n \;=\; \frac{1}{1-x},$$

  with the radius of convergence 1 (the series converges when $|x| < 1$).

Operations on egf's:

- $\left(\sum_{n=0}^{\infty} \frac{a_n x^n}{n!}\right) \pm \left(\sum_{n=0}^{\infty} \frac{b_n x^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{(a_n \pm b_n) x^n}{n!}$

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Example 2.6

Let the sequence $\{d_n\}_{n=0}^{\infty}$ be defined recursively as follows:

- $d_0 = 1,\ d_1 = 0;$
- $d_{n+2} = (n+1)(d_n + d_{n+1})$ for all integers $n \geq 0$.

Find a closed formula for the egf of the sequences $\{d_n\}_{n=0}^{\infty}$, and then find a non-recursive formula for d_n.

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- Remark: d_n is the number of "derangements" of an n-element set, i.e. the number of permutations of $\{1, \ldots, n\}$ with no fixed points. (details: Lecture Notes.)

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Solution. Let $d(x) = \sum_{n=0}^{\infty} \frac{d_n x^n}{n!}$ be the egf of the sequence $\{d_n\}_{n=0}^{\infty}$. We first differentiate $d(x)$, and then we apply the recursive formula, as follows.

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Solution (continued).

$$\begin{aligned}
d'(x) &= \sum_{n=0}^{\infty} \frac{d_{n+1}x^n}{n!} \\
&= \sum_{n=1}^{\infty} \frac{d_{n+1}x^n}{n!} && \text{because } d_1 = 0 \\
&= \left(\sum_{n=1}^{\infty} \frac{nd_{n-1}x^n}{n!}\right) + \left(\sum_{n=1}^{\infty} \frac{nd_nx^n}{n!}\right) && \text{by the recursive formula} \\
&= x\left(\sum_{n=0}^{\infty} \frac{d_nx^n}{n!}\right) + x\left(\sum_{n=0}^{\infty} \frac{d_{n+1}x^n}{n!}\right) \\
&= xd(x) + xd'(x).
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By integrating both sides, we get

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Solution (continued). Reminder: $\ln(d(x)) = -\ln(1-x) - x$.
By exponentiating both sides, we get

$$d(x) \;=\; \tfrac{e^{-x}}{1-x}.$$

We have now obtained a closed formula for the exponential generating function $d(x)$.

- $\left(\sum_{n=0}^{\infty} \frac{a_n x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{b_n x^n}{n!}\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\right) \frac{x^n}{n!}$

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Solution (continued). Reminder: $d(x) = \frac{e^{-x}}{1-x}$.
To obtain a formula for d_n, we note that

- $e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$;
- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \frac{n! x^n}{n!}$.

By the formula for the product of egf's, we now have that, for all integers $n \geq 0$, $d_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)!$, and we are done.