NDMI012: Combinatorics and Graph Theory 2

Lecture #11

Burnside's lemma and applications

Irena Penev

May 3, 2022

Definition

A *group* is a set G, together with a binary operation $\circ$, satisfying the following properties:

- $\circ$ is associative, i.e. for all $g_1, g_2, g_3 \in G$, $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$;
- there exists some $e \in G$, called the *identity element*, s.t. for all $g \in G$, $e \circ g = g \circ e = g$;
- for all $g \in G$, there exists some $g' \in G$, called the *inverse* of g, s.t. $g \circ g' = g' \circ g = e$.

Definition

A *group* is a set G, together with a binary operation $\circ$, satisfying the following properties:

- $\circ$ is associative, i.e. for all $g_1, g_2, g_3 \in G$, $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$;
- there exists some $e \in G$, called the *identity element*, s.t. for all $g \in G$, $e \circ g = g \circ e = g$;
- for all $g \in G$, there exists some $g' \in G$, called the *inverse* of g, s.t. $g \circ g' = g' \circ g = e$.

- Usually, for $g_1, g_2 \in G$, we write "$g_1 g_2$" instead of "$g_1 \circ g_2$."

Definition

A *group* is a set G, together with a binary operation $\circ$, satisfying the following properties:

- $\circ$ is associative, i.e. for all $g_1, g_2, g_3 \in G$, $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$;
- there exists some $e \in G$, called the *identity element*, s.t. for all $g \in G$, $e \circ g = g \circ e = g$;
- for all $g \in G$, there exists some $g' \in G$, called the *inverse* of g, s.t. $g \circ g' = g' \circ g = e$.

- Usually, for $g_1, g_2 \in G$, we write "$g_1 g_2$" instead of "$g_1 \circ g_2$."
- The identity element is unique.
  - Indeed, suppose e_1, e_2 are identity elements of G. Then $e_1 e_2 = e_1$ (because e_2 is an identity element), and $e_1 e_2 = e_2$ (because e_1 is an identity element). So, $e_1 = e_2$.

Definition

A *group* is a set G, together with a binary operation $\circ$, satisfying the following properties:

- $\circ$ is associative, i.e. for all $g_1, g_2, g_3 \in G$, $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$;
- there exists some $e \in G$, called the *identity element*, s.t. for all $g \in G$, $e \circ g = g \circ e = g$;
- for all $g \in G$, there exists some $g' \in G$, called the *inverse* of g, s.t. $g \circ g' = g' \circ g = e$.

- Usually, for $g_1, g_2 \in G$, we write "$g_1 g_2$" instead of "$g_1 \circ g_2$."
- The identity element is unique.
  - Indeed, suppose e_1, e_2 are identity elements of G. Then $e_1 e_2 = e_1$ (because e_2 is an identity element), and $e_1 e_2 = e_2$ (because e_1 is an identity element). So, $e_1 = e_2$.
  - Typically, this identity element is denoted by 1_G, or simply 1.

Definition

A *group* is a set G, together with a binary operation $\circ$, satisfying the following properties:

- $\circ$ is associative, i.e. for all $g_1, g_2, g_3 \in G$, $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$;
- there exists some $e \in G$, called the *identity element*, s.t. for all $g \in G$, $e \circ g = g \circ e = g$;
- for all $g \in G$, there exists some $g' \in G$, called the *inverse* of g, s.t. $g \circ g' = g' \circ g = e$.

- It can be shown that each element of G has a unique inverse.

Definition

A *group* is a set G, together with a binary operation $\circ$, satisfying the following properties:

- $\circ$ is associative, i.e. for all $g_1, g_2, g_3 \in G$, $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$;
- there exists some $e \in G$, called the *identity element*, s.t. for all $g \in G$, $e \circ g = g \circ e = g$;
- for all $g \in G$, there exists some $g' \in G$, called the *inverse* of g, s.t. $g \circ g' = g' \circ g = e$.

- It can be shown that each element of G has a unique inverse.
  - Indeed, if $g_1, g_2 \in G$ are both inverses of some $g \in G$, then

$$
\begin{aligned}
g_1 &= g_1 1_G && \text{because } 1_G \text{ is the identity element} \\
&= g_1(gg_2) && \text{because } g_2 \text{ is an inverse of } g \\
&= (g_1 g)g_2 && \text{because } \circ \text{ is associative} \\
&= 1_G g_2 && \text{because } g_1 \text{ is an inverse of } g \\
&= g_2 && \text{because } 1_G \text{ is an identity element}
\end{aligned}
$$

Definition

A *group* is a set G, together with a binary operation $\circ$, satisfying the following properties:

- $\circ$ is associative, i.e. for all $g_1, g_2, g_3 \in G$, $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$;
- there exists some $e \in G$, called the *identity element*, s.t. for all $g \in G$, $e \circ g = g \circ e = g$;
- for all $g \in G$, there exists some $g' \in G$, called the *inverse* of g, s.t. $g \circ g' = g' \circ g = e$.

- It can be shown that each element of G has a unique inverse.
  - Indeed, if $g_1, g_2 \in G$ are both inverses of some $g \in G$, then

$$\begin{aligned} g_1 &= g_1 1_G && \text{because } 1_G \text{ is the identity element} \\ &= g_1(gg_2) && \text{because } g_2 \text{ is an inverse of } g \\ &= (g_1 g)g_2 && \text{because } \circ \text{ is associative} \\ &= 1_G g_2 && \text{because } g_1 \text{ is an inverse of } g \\ &= g_2 && \text{because } 1_G \text{ is an identity element} \end{aligned}$$

  - The inverse of g is denoted by g^{-1}.

- For a set X, $\mathrm{Sym}(X)$ is the group of all permutations of X; the group operation is the composition of functions, and the identity element is Id_X.

- For a set X, $\mathrm{Sym}(X)$ is the group of all permutations of X; the group operation is the composition of functions, and the identity element is Id_X.
- For a positive integer n, the group of permutations of the set $\{1, \ldots, n\}$ is denoted by $\mathrm{Sym}(n)$ or Sym_n.

- For a set X, $\mathrm{Sym}(X)$ is the group of all permutations of X; the group operation is the composition of functions, and the identity element is Id_X.
- For a positive integer n, the group of permutations of the set $\{1, \ldots, n\}$ is denoted by $\mathrm{Sym}(n)$ or Sym_n.
- A permutation $\pi \in \mathrm{Sym}(n)$ can be denoted by

$$\left(\begin{array}{cccc} 1 & 2 & \ldots & n \\ \pi(1) & \pi(2) & \ldots & \pi(n) \end{array} \right).$$

- Recall that each permutation in $\mathrm{Sym}(n)$ can be represented as a composition of disjoint cycles.

- Recall that each permutation in Sym(n) can be represented as a composition of disjoint cycles.
- For example, the following permutation in Sym(5)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix}$$

can be represented as (143)(25).

- Recall that each permutation in Sym(n) can be represented as a composition of disjoint cycles.
- For example, the following permutation in Sym(5)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix}$$

  can be represented as (143)(25).
- Cycles of length one are usually omitted (when n is clear from context).
  - For example, in Sym(5), instead of (124)(3)(5), we typically write simply (124).

Definition

A *left action* (or simply *action*) of a group G on a set X is a function $a : G \times X \to X$ that satisfies the following two properties:

- for all $x \in X$, $a(1_G, x) = x$.
- for all $g_1, g_2 \in G$ and $x \in X$, $a(g_1, a(g_2, x)) = a(g_1 g_2, x)$.

Definition

A *left action* (or simply *action*) of a group G on a set X is a function $a : G \times X \to X$ that satisfies the following two properties:

- for all $x \in X$, $a(1_G, x) = x$.
- for all $g_1, g_2 \in G$ and $x \in X$, $a(g_1, a(g_2, x)) = a(g_1 g_2, x)$.

- Often, instead of $a(g, x)$, we write simply $g \cdot x$.

Definition

A *left action* (or simply *action*) of a group G on a set X is a function $a : G \times X \to X$ that satisfies the following two properties:

- for all $x \in X$, $a(1_G, x) = x$.
- for all $g_1, g_2 \in G$ and $x \in X$, $a(g_1, a(g_2, x)) = a(g_1 g_2, x)$.

- Often, instead of $a(g, x)$, we write simply $g \cdot x$.
- So, using this notation, the axioms above become:
    - for all $x \in X$, $1_G \cdot x = x$.
    - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.

Definition

A *left action* (or simply *action*) of a group G on a set X is a function $a : G \times X \to X$ that satisfies the following two properties:

- for all $x \in X$, $a(1_G, x) = x$.
- for all $g_1, g_2 \in G$ and $x \in X$, $a(g_1, a(g_2, x)) = a(g_1 g_2, x)$.

- Often, instead of $a(g, x)$, we write simply $g \cdot x$.
- So, using this notation, the axioms above become:
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Note that these axioms imply that, for all $g \in G$ and $x, y \in X$, if $g \cdot x = y$, then $g^{-1} \cdot y = x$.

Definition

A *left action* (or simply *action*) of a group G on a set X is a function $a : G \times X \to X$ that satisfies the following two properties:

- for all $x \in X$, $a(1_G, x) = x$.
- for all $g_1, g_2 \in G$ and $x \in X$, $a(g_1, a(g_2, x)) = a(g_1 g_2, x)$.

- Often, instead of $a(g, x)$, we write simply $g \cdot x$.
- So, using this notation, the axioms above become:
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Note that these axioms imply that, for all $g \in G$ and $x, y \in X$, if $g \cdot x = y$, then $g^{-1} \cdot y = x$.
  - Indeed, if $g \cdot x = y$, then $g^{-1} \cdot y = g^{-1} \cdot (g \cdot x) = (g^{-1} g) \cdot x = 1_G \cdot x = x$.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.

Example 2.1

Any group G acts on itself in a natural way: for all $g \in G$ and $x \in G$,[a] we set $g \cdot x = gx$.

[a] Here, $X = G$.

- Given an action $a : G \times X \to X$ of a group G on a set X, and an element $g \in G$, we define a function $a_g : X \to X$ by setting $a_g(x) = a(g, x)$ for all $x \in X$.

- Given an action $a : G \times X \to X$ of a group G on a set X, and an element $g \in G$, we define a function $a_g : X \to X$ by setting $a_g(x) = a(g, x)$ for all $x \in X$.

Proposition 2.2

Let $a : G \times X \to X$ be an action of a group G on a set X. Then for all $g \in G$, the function a_g is a permutation of X.

Proof.

- Given an action $a : G \times X \to X$ of a group G on a set X, and an element $g \in G$, we define a function $a_g : X \to X$ by setting $a_g(x) = a(g, x)$ for all $x \in X$.

Proposition 2.2

Let $a : G \times X \to X$ be an action of a group G on a set X. Then for all $g \in G$, the function a_g is a permutation of X.

Proof. Fix $g \in G$, and consider its inverse g^{-1}. Then for all $x \in X$, we have that

$$
\begin{aligned}
a_{g^{-1}} \circ a_g(x) &= a(g^{-1}, a(g, x)) \\
&= a(g^{-1}g, x) \\
&= a(1_G, x) \\
&= x,
\end{aligned}
$$

and so $a_{g^{-1}} \circ a_g = \mathrm{Id}_X$.

- Given an action $a : G \times X \to X$ of a group G on a set X, and an element $g \in G$, we define a function $a_g : X \to X$ by setting $a_g(x) = a(g, x)$ for all $x \in X$.

Proposition 2.2

Let $a : G \times X \to X$ be an action of a group G on a set X. Then for all $g \in G$, the function a_g is a permutation of X.

Proof. Fix $g \in G$, and consider its inverse g^{-1}. Then for all $x \in X$, we have that

$$\begin{aligned} a_{g^{-1}} \circ a_g(x) &= a(g^{-1}, a(g, x)) \\ &= a(g^{-1}g, x) \\ &= a(1_G, x) \\ &= x, \end{aligned}$$

and so $a_{g^{-1}} \circ a_g = \mathrm{Id}_X$. A completely analogous argument shows that $a_g \circ a_{g^{-1}} = \mathrm{Id}_X$.

- Given an action $a : G \times X \to X$ of a group G on a set X, and an element $g \in G$, we define a function $a_g : X \to X$ by setting $a_g(x) = a(g, x)$ for all $x \in X$.

Proposition 2.2

Let $a : G \times X \to X$ be an action of a group G on a set X. Then for all $g \in G$, the function a_g is a permutation of X.

Proof. Fix $g \in G$, and consider its inverse g^{-1}. Then for all $x \in X$, we have that

$$\begin{aligned} a_{g^{-1}} \circ a_g(x) &= a(g^{-1}, a(g, x)) \\ &= a(g^{-1}g, x) \\ &= a(1_G, x) \\ &= x, \end{aligned}$$

and so $a_{g^{-1}} \circ a_g = \mathrm{Id}_X$. A completely analogous argument shows that $a_g \circ a_{g^{-1}} = \mathrm{Id}_X$. So, $a_g : X \to X$ is a bijection with inverse $a_{g^{-1}}$, and the result follows.

- Given an action $a : G \times X \to X$ of a group G on a set X, and an element $g \in G$, we define a function $a_g : X \to X$ by setting $a_g(x) = a(g, x)$ for all $x \in X$.

Proposition 2.2

Let $a : G \times X \to X$ be an action of a group G on a set X. Then for all $g \in G$, the function a_g is a permutation of X.

- A converse of sorts of Proposition 2.2 also holds: for any set X and any permutation π of X, there is a group G, an action a of G on X, and an element $g \in G$ s.t. $a_g = \pi$.

- Given an action $a : G \times X \to X$ of a group G on a set X, and an element $g \in G$, we define a function $a_g : X \to X$ by setting $a_g(x) = a(g, x)$ for all $x \in X$.

Proposition 2.2

Let $a : G \times X \to X$ be an action of a group G on a set X. Then for all $g \in G$, the function a_g is a permutation of X.

- A converse of sorts of Proposition 2.2 also holds: for any set X and any permutation π of X, there is a group G, an action a of G on X, and an element $g \in G$ s.t. $a_g = \pi$.
  - Indeed, for fixed X and π, we set $G := \mathrm{Sym}(X)$ (the group operation is the composition of functions), we define $a : G \times X \to X$ by $(\sigma, x) \mapsto \sigma(x)$, and we set $g := \pi$.

- Given an action $a : G \times X \to X$ of a group G on a set X, and an element $g \in G$, we define a function $a_g : X \to X$ by setting $a_g(x) = a(g, x)$ for all $x \in X$.

Proposition 2.2

Let $a : G \times X \to X$ be an action of a group G on a set X. Then for all $g \in G$, the function a_g is a permutation of X.

- A converse of sorts of Proposition 2.2 also holds: for any set X and any permutation π of X, there is a group G, an action a of G on X, and an element $g \in G$ s.t. $a_g = \pi$.
  - Indeed, for fixed X and π, we set $G := \text{Sym}(X)$ (the group operation is the composition of functions), we define $a : G \times X \to X$ by $(\sigma, x) \mapsto \sigma(x)$, and we set $g := \pi$.
  - Then for all $x \in X$, we have that $a_g(x) = a_\pi(x) = a(\pi, x) = \pi(x)$, and so $a_g = \pi$.

- Given an action $a : G \times X \to X$ of a group G on a set X, and an element $g \in G$, we define a function $a_g : X \to X$ by setting $a_g(x) = a(g, x)$ for all $x \in X$.

Proposition 2.2

Let $a : G \times X \to X$ be an action of a group G on a set X. Then for all $g \in G$, the function a_g is a permutation of X.

- A converse of sorts of Proposition 2.2 also holds: for any set X and any permutation π of X, there is a group G, an action a of G on X, and an element $g \in G$ s.t. $a_g = \pi$.
  - Indeed, for fixed X and π, we set $G := \text{Sym}(X)$ (the group operation is the composition of functions), we define $a : G \times X \to X$ by $(\sigma, x) \mapsto \sigma(x)$, and we set $g := \pi$.
  - Then for all $x \in X$, we have that $a_g(x) = a_\pi(x) = a(\pi, x) = \pi(x)$, and so $a_g = \pi$.
- So, the study of group actions is essentially the same as the study of set permutations.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.

Example 2.3

Consider a cube in $\mathbb{R}^3$, and let R_{cube} be the group of rotations of $\mathbb{R}^3$ that map this cube to itself. (Here, the group operation is the composition of functions/rotations, and the identity element is the identity function on $\mathbb{R}^3$.) The group R_{cube} acts on the faces of the cube in a natural way: for each rotation $r \in R_{\text{cube}}$ and each face f of the cube, $r \cdot f$ is the face of the cube to which the rotation r maps/moves the face f.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.

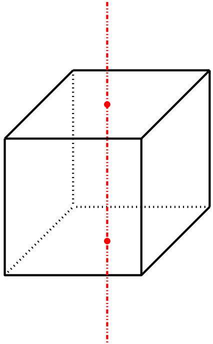
Example 2.3

Consider a cube in $\mathbb{R}^3$, and let R_{cube} be the group of rotations of $\mathbb{R}^3$ that map this cube to itself. (Here, the group operation is the composition of functions/rotations, and the identity element is the identity function on $\mathbb{R}^3$.) The group R_{cube} acts on the faces of the cube in a natural way: for each rotation $r \in R_{\text{cube}}$ and each face f of the cube, $r \cdot f$ is the face of the cube to which the rotation r maps/moves the face f.

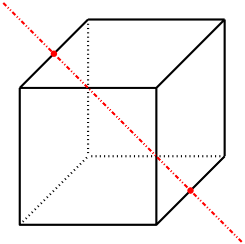
- Note that $|R_{\text{cube}}| = 24$.
- Indeed, let's check what the rotations can be!

- Rotations in R_{cube}:
  1. the identity function;

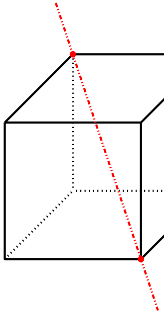
- Rotations in R_{cube}:
  1. the identity function;
  2. nine rotations about an axis passing though centers of opposite faces of the cube (there are three choices of axis, and for each choice, we can rotate by 90°, 180°, or 270°);



- Rotations in R_{cube}:
    3. six rotations about an axis passing through centers of opposite edges of the cube (there are six choices of axis, and for each choice, we can rotate only by 180°);



- Rotations in R_{cube}:
  4. eight rotations around axes passing through opposite vertices of the cube (there are four choices of axis, and in each case, we can rotate by 120° or 240°).



- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.

Definition

Suppose that a is an action of a group G on a set X. Then

- for each $g \in G$, a *fixed point* of g is any $x \in X$ s.t. $g \cdot x = x$, and we set $X^g := \{x \in X \mid g \cdot x = x\}$;[a]
- for each $x \in X$, we define the *stabilizer* of x to be $G_x := \{g \in G \mid g \cdot x = x\}$;
- for each $x \in X$, we define the *orbit* of x to be $G \cdot x := \{g \cdot x \mid g \in G\}$.[b]

---

[a] So, X^g is the set of all fixed points of g (with respect to the action a).

[b] So, the orbit of x is the set of all elements of x that G can "move" x to.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$.

Proposition 2.4

Let a be an action of a group G on a set X. Then

- for all $x \in X$, we have that $x \in G \cdot x$;
- the orbits of the action a form a partition of X.

Proof.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$.

Proposition 2.4

Let a be an action of a group G on a set X. Then

- for all $x \in X$, we have that $x \in G \cdot x$;
- the orbits of the action a form a partition of X.

Proof. First, since $1_G \cdot x = x$, we see that $x \in G \cdot x$.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$.

Proposition 2.4

Let a be an action of a group G on a set X. Then

- for all $x \in X$, we have that $x \in G \cdot x$;
- the orbits of the action a form a partition of X.

Proof. First, since $1_G \cdot x = x$, we see that $x \in G \cdot x$. In particular, each element of X belongs to some orbit.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$.

Proposition 2.4

Let a be an action of a group G on a set X. Then

- for all $x \in X$, we have that $x \in G \cdot x$;
- the orbits of the action a form a partition of X.

Proof. First, since $1_G \cdot x = x$, we see that $x \in G \cdot x$. In particular, each element of X belongs to some orbit.

It remains to show that any two distinct orbits are disjoint.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$.

Proof (continued). So, fix $x_1, x_2 \in X$; we must show that $G \cdot x_1$ and $G \cdot x_2$ are either equal or disjoint.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$.

Proof (continued). So, fix $x_1, x_2 \in X$; we must show that $G \cdot x_1$ and $G \cdot x_2$ are either equal or disjoint. Suppose that $G \cdot x_1$ and $G \cdot x_2$ are not disjoint; we claim that $G \cdot x_1 = G \cdot x_2$.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$.

Proof (continued). So, fix $x_1, x_2 \in X$; we must show that $G \cdot x_1$ and $G \cdot x_2$ are either equal or disjoint. Suppose that $G \cdot x_1$ and $G \cdot x_2$ are not disjoint; we claim that $G \cdot x_1 = G \cdot x_2$. We will show that $G \cdot x_1 \subseteq G \cdot x_2$; the proof of the reverse inclusion is analogous.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$.

Proof (continued). So, fix $x_1, x_2 \in X$; we must show that $G \cdot x_1$ and $G \cdot x_2$ are either equal or disjoint. Suppose that $G \cdot x_1$ and $G \cdot x_2$ are not disjoint; we claim that $G \cdot x_1 = G \cdot x_2$. We will show that $G \cdot x_1 \subseteq G \cdot x_2$; the proof of the reverse inclusion is analogous.

Fix some $y \in (G \cdot x_1) \cap (G \cdot x_2)$.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$.

Proof (continued). So, fix $x_1, x_2 \in X$; we must show that $G \cdot x_1$ and $G \cdot x_2$ are either equal or disjoint. Suppose that $G \cdot x_1$ and $G \cdot x_2$ are not disjoint; we claim that $G \cdot x_1 = G \cdot x_2$. We will show that $G \cdot x_1 \subseteq G \cdot x_2$; the proof of the reverse inclusion is analogous.

Fix some $y \in (G \cdot x_1) \cap (G \cdot x_2)$. Then there exist $g_1, g_2 \in G$ s.t. $y = g_1 \cdot x_1$ and $y = g_2 \cdot x_2$; so, $g_1 \cdot x_1 = g_2 \cdot x_2$.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$.

Proof (continued). So, fix $x_1, x_2 \in X$; we must show that $G \cdot x_1$ and $G \cdot x_2$ are either equal or disjoint. Suppose that $G \cdot x_1$ and $G \cdot x_2$ are not disjoint; we claim that $G \cdot x_1 = G \cdot x_2$. We will show that $G \cdot x_1 \subseteq G \cdot x_2$; the proof of the reverse inclusion is analogous.

Fix some $y \in (G \cdot x_1) \cap (G \cdot x_2)$. Then there exist $g_1, g_2 \in G$ s.t. $y = g_1 \cdot x_1$ and $y = g_2 \cdot x_2$; so, $g_1 \cdot x_1 = g_2 \cdot x_2$. But then $x_1 = (g_1^{-1} g_2) \cdot x_2$.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$.

Proof (continued). So, fix $x_1, x_2 \in X$; we must show that $G \cdot x_1$ and $G \cdot x_2$ are either equal or disjoint. Suppose that $G \cdot x_1$ and $G \cdot x_2$ are not disjoint; we claim that $G \cdot x_1 = G \cdot x_2$. We will show that $G \cdot x_1 \subseteq G \cdot x_2$; the proof of the reverse inclusion is analogous.

Fix some $y \in (G \cdot x_1) \cap (G \cdot x_2)$. Then there exist $g_1, g_2 \in G$ s.t. $y = g_1 \cdot x_1$ and $y = g_2 \cdot x_2$; so, $g_1 \cdot x_1 = g_2 \cdot x_2$. But then $x_1 = (g_1^{-1} g_2) \cdot x_2$. Now, for all $g \in G$, we have that

$$g \cdot x_1 \;=\; g \cdot ((g_1^{-1} g_2) \cdot x_2) \;=\; (g g_1^{-1} g_2) \cdot x_2,$$

and so $g \cdot x_1 \in G \cdot x_2$.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$.

Proof (continued). So, fix $x_1, x_2 \in X$; we must show that $G \cdot x_1$ and $G \cdot x_2$ are either equal or disjoint. Suppose that $G \cdot x_1$ and $G \cdot x_2$ are not disjoint; we claim that $G \cdot x_1 = G \cdot x_2$. We will show that $G \cdot x_1 \subseteq G \cdot x_2$; the proof of the reverse inclusion is analogous.

Fix some $y \in (G \cdot x_1) \cap (G \cdot x_2)$. Then there exist $g_1, g_2 \in G$ s.t. $y = g_1 \cdot x_1$ and $y = g_2 \cdot x_2$; so, $g_1 \cdot x_1 = g_2 \cdot x_2$. But then $x_1 = (g_1^{-1} g_2) \cdot x_2$. Now, for all $g \in G$, we have that

$$g \cdot x_1 \quad = \quad g \cdot ((g_1^{-1} g_2) \cdot x_2) \quad = \quad (g g_1^{-1} g_2) \cdot x_2,$$

and so $g \cdot x_1 \in G \cdot x_2$. Thus, $G \cdot x_1 \subseteq G \cdot x_2$, and we are done.

- Group action axioms (group G acting on set X):
    - for all $x \in X$, $1_G \cdot x = x$.
    - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$.

Proposition 2.4

Let a be an action of a group G on a set X. Then

- for all $x \in X$, we have that $x \in G \cdot x$;
- the orbits of the action a form a partition of X.

- Let X/G be the partition of X into orbits (of a).
- So, $|X/G|$ is the number of orbits of the action a.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Set of fixed points of $g \in G$: $X^g := \{x \in X \mid g \cdot x = x\}$;
- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;
- Partition of X into orbits of action a: X/G.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Set of fixed points of $g \in G$: $X^g := \{x \in X \mid g \cdot x = x\}$;
- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;
- Partition of X into orbits of action a: X/G.

Burnside's lemma

Let a be an action of a finite group G on a finite set X. Then

$$|X/G| \quad = \quad \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Set of fixed points of $g \in G$: $X^g := \{x \in X \mid g \cdot x = x\}$;
- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;
- Partition of X into orbits of action a: X/G.

Burnside's lemma

Let a be an action of a finite group G on a finite set X. Then

$$|X/G| \quad = \quad \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

- So, Burnside's lemma states (roughly) that the number of orbits is equal to the average number of fixed points.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Set of fixed points of $g \in G$: $X^g := \{x \in X \mid g \cdot x = x\}$;
- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;
- Partition of X into orbits of action a: X/G.

Burnside's lemma

Let a be an action of a finite group G on a finite set X. Then

$$|X/G| \quad = \quad \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

- So, Burnside's lemma states (roughly) that the number of orbits is equal to the average number of fixed points.
- We'll prove Burnside's lemma later.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Set of fixed points of $g \in G$: $X^g := \{x \in X \mid g \cdot x = x\}$;
- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;
- Partition of X into orbits of action a: X/G.

Burnside's lemma

Let a be an action of a finite group G on a finite set X. Then

$$|X/G| \;=\; \frac{1}{|G|}\sum_{g \in G} |X^g|.$$

- So, Burnside's lemma states (roughly) that the number of orbits is equal to the average number of fixed points.
- We'll prove Burnside's lemma later.
- First, let's look at some applications.

- Burnside's lemma: $|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$.

- Burnside's lemma: $|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$.

Example 3.1

Let R_{cube} be the group of rotations of the cube, as in Example 2.3, and let k be a positive integer. Let B_k be the set of all colorings of the faces of the cube using the color set $\{1,\ldots,k\}$. Then R_{cube} acts on the set B_k in the natural way: a rotation $r \in R_{\text{cube}}$ maps each element of B_k to an appropriately rotated coloring. Two colorings of the cube are *equivalent* if one can be transformed into the other by a rotation in R_{cube}. Compute the number of non-equivalent colorings of the cube using the color set $\{1,\ldots,k\}$.

- Burnside's lemma: $|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$.

Solution.

- Burnside's lemma: $|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$.

Solution. Two colorings of the cube using the color set $\{1,\ldots,k\}$ are equivalent iff they belong to the same orbit of our group action.

- Burnside's lemma: $|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$.

Solution. Two colorings of the cube using the color set $\{1,\ldots,k\}$ are equivalent iff they belong to the same orbit of our group action. So, the number of non-equivalent colorings of the cube using the color set $\{1,\ldots,k\}$ is precisely equal to the number of orbits of our action of R_{cube} on B_k, which we will compute using Burnside's lemma.

- Burnside's lemma: $|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$.

Solution. Two colorings of the cube using the color set $\{1,\ldots,k\}$ are equivalent iff they belong to the same orbit of our group action. So, the number of non-equivalent colorings of the cube using the color set $\{1,\ldots,k\}$ is precisely equal to the number of orbits of our action of R_{cube} on B_k, which we will compute using Burnside's lemma.

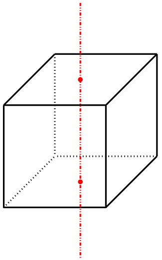
We know that $|R_{\text{cube}}| = 24$ (see Example 2.3), and for each $r \in R_{\text{cube}}$, we compute B_k^r as follows.

Solution (continued).

- If r is the identity rotation, then $|B_k^r| = |B_k| = k^6$.

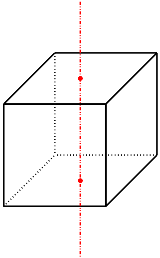
Solution (continued).

- If r is the identity rotation, then $|B_k^r| = |B_k| = k^6$.
- If r is a rotation by 90° or 270° about an axis passing though the center of opposite faces (there are a total of six such r's), then r fixes precisely the colorings in which the faces not pierced by the axis have the same color. So, we choose one of k colors for one of the faces pieced by the axis, one of k colors for the other face pierced by the axis, and one of k colors for all the remaining four faces. In total, we get $|B_k^r| = k^3$.



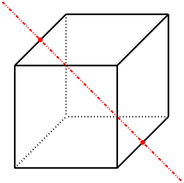
Solution (continued).

- If r is a rotation by 180° about an axis passing though the center of opposite faces (there are a total of three such r's), then r fixes exactly the the colorings for which the opposite faces that are not pierced by the axis have the same color. There are two pairs of opposite faces not pierced by our axis, and it follows that $|B_k^r| = k^4$.



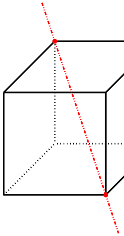
Solution (continued).

- If r is a rotation by 180° about an axis passing though the center of opposite edges (there are a total of six such r's), then r fixes exactly the colorings for which the two opposite faces not incident with the edges pierced by the axis have the same color, and in which, for each pierced edge, the two faces incident with this edge have the same color. So, $|B_k^r| = k^3$.



Solution (continued).

- Finally, if r is a rotation by 120° or 240° about an axis passing though opposite vertices (there are a total of eight such r's), then r fixes exactly the colorings for which the three incident faces with each of the pierced vertices have the same color. So, $|B_k^r| = k^2$.



- Burnside's lemma: $|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$.

Solution (continued).

- Burnside's lemma: $|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$.

Solution (continued). So, by Burnside's lemma and Example 3.1, the total number of orbits of our action (and therefore, the total number of non-equivalent colorings) is

$$\begin{aligned}
\frac{1}{|R_{\text{cube}}|}\sum_{r\in R_{\text{cube}}}|B_k^r| &= \frac{k^6+6k^3+3k^4+6k^3+8k^2}{24} \\
&= \frac{k^6+3k^4+12k^3+8k^2}{24}
\end{aligned}$$

- Burnside's lemma: $|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$.

- Burnside's lemma: $|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$.

Example 3.2

Find the number of non-isomorphic graphs on five vertices.

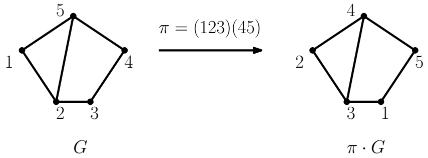
Solution.

- Burnside's lemma: $|X/G| = \frac{1}{|G|}\sum_{g\in G} |X^g|$.

Example 3.2

Find the number of non-isomorphic graphs on five vertices.

Solution. Let X be the set of all graphs on the vertex set $\{1,\ldots,5\}$. We let $\mathrm{Sym}(5)$ act on X in the natural way: given a graph $G \in X$ and a permutation $\pi \in \mathrm{Sym}(5)$, we let $\pi \cdot G$ be the graph with vertex set $\{1,\ldots,5\}$, in which distinct vertices $i, j \in \{1,\ldots,5\}$ are adjacent iff $\pi^{-1}(i)$ and $\pi^{-1}(j)$ are adjacent in G.



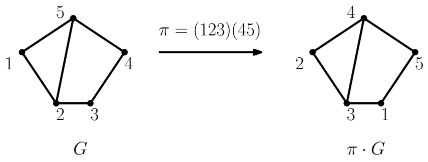
(Equivalently: $\pi \cdot G$ has the same vertex set as G; each edge ij of G turns into an edge $\pi(i)\pi(j)$ of $\pi \cdot G$; and each non-edge ij of G turns into a non-edge $\pi(i)\pi(j)$ of $\pi \cdot G$.)

- Burnside's lemma: $|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$.

Example 3.2

Find the number of non-isomorphic graphs on five vertices.

Solution (continued).



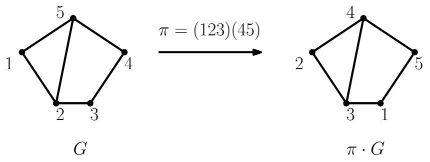
Clearly, two graphs in X are isomorphic iff they belong to the same orbit of this action.

- Burnside's lemma: $|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$.

Example 3.2

Find the number of non-isomorphic graphs on five vertices.

Solution (continued).



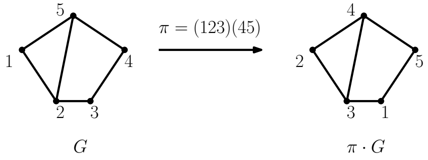
Clearly, two graphs in X are isomorphic iff they belong to the same orbit of this action. So, the number of non-isomorphic graphs on five vertices is equal to the number of orbits of our action.

- Burnside's lemma: $|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$.

Example 3.2

Find the number of non-isomorphic graphs on five vertices.

Solution (continued).



Clearly, two graphs in X are isomorphic iff they belong to the same orbit of this action. So, the number of non-isomorphic graphs on five vertices is equal to the number of orbits of our action. We will compute the number of orbits using Burnside's lemma.

- Burnside's lemma: $|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$.

Example 3.2

Find the number of non-isomorphic graphs on five vertices.

Solution (continued). Clearly, $|\mathrm{Sym}(5)| = 5!$.

- Burnside's lemma: $|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$.

Example 3.2

Find the number of non-isomorphic graphs on five vertices.

Solution (continued). Clearly, $|\mathrm{Sym}(5)| = 5!$. We compute the number of fixed points of a permutation $\pi \in \mathrm{Sym}(5)$ according to the cycle structure of π.

- Burnside's lemma: $|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$.

Example 3.2

Find the number of non-isomorphic graphs on five vertices.

Solution (continued). Clearly, $|\mathrm{Sym}(5)| = 5!$. We compute the number of fixed points of a permutation $\pi \in \mathrm{Sym}(5)$ according to the cycle structure of π.

- If π is the identity function, then π fixes all elements of X, i.e. $|X^\pi| = |X| = 2^{\binom{5}{2}} = 2^{10}$.

- Burnside's lemma: $|X/G| = \frac{1}{|G|}\sum_{g \in G} |X^g|$.

Example 3.2

Find the number of non-isomorphic graphs on five vertices.

Solution (continued). Clearly, $|\text{Sym}(5)| = 5!$. We compute the number of fixed points of a permutation $\pi \in \text{Sym}(5)$ according to the cycle structure of π.

- If π is the identity function, then π fixes all elements of X, i.e. $|X^\pi| = |X| = 2^{\binom{5}{2}} = 2^{10}$.
- If $\pi = (ab)$, for distinct $a, b \in \{1, \dots, 5\}$ (note: there are $\binom{5}{2} = 10$ such π's), then π fixes precisely the graphs $G \in X$ s.t. $N_G(a) \setminus \{b\} = N_G(b) \setminus \{a\}$. So, we can freely select the neighbors of a (the neighbors of b are then forced), and we can choose adjacency between vertices in $\{1, \dots, 5\} \setminus \{a, b\}$ arbitrarily. There are 2^4 ways to choose the neighbors of a, and there are $2^{\binom{3}{2}} = 2^3$ ways to choose adjacency between vertices in $\{1, \dots, 5\} \setminus \{a, b\}$. So, $|X^\pi| = 2^4 \cdot 2^3 = 2^7$.

Example 3.2

Find the number of non-isomorphic graphs on five vertices.

Solution (continued).

- If $\pi = (ab)(cd)$ for pairwise distinct $a, b, c, d \in \{1, \dots, 5\}$ (note: there are 15 such π's), then π fixes precisely the graphs $G \in X$ satisfying the following three properties:
    - ac is an edge iff bd is an edge,
    - ad is an edge iff bc is an edge,
    - the fifth vertex of G (i.e. the unique vertex in $\{1, \dots, 5\} \setminus \{a, b, c, d\}$) is adjacent to a iff it is adjacent to b, and is adjacent to c iff it is adjacent to d.

  So, $|X^\pi| = 2^6$.

Example 3.2

Find the number of non-isomorphic graphs on five vertices.

Solution (continued).

- If $\pi = (abc)$, for pairwise distinct $a, b, c \in \{1, \dots, 5\}$ (note: there are 20 such π's), then π fixes precisely the graphs $G \in X$ in which $\{a, b, c\}$ is either a clique or a stable set, and each of the remaining two vertices (i.e. vertices in $\{1, \dots, 5\} \setminus \{a, b, c\}$) is either complete or anticomplete to $\{a, b, c\}$. So, $|X^\pi| = 2^4$.

Example 3.2

Find the number of non-isomorphic graphs on five vertices.

Solution (continued).

- If $\pi = (abc)$, for pairwise distinct $a, b, c \in \{1, \ldots, 5\}$ (note: there are 20 such π's), then π fixes precisely the graphs $G \in X$ in which $\{a, b, c\}$ is either a clique or a stable set, and each of the remaining two vertices (i.e. vertices in $\{1, \ldots, 5\} \setminus \{a, b, c\}$) is either complete or anticomplete to $\{a, b, c\}$. So, $|X^\pi| = 2^4$.
- If $\pi = (abc)(de)$, for pairwise distinct $a, b, c, d, e \in \{1, \ldots, 5\}$ (note: there are 20 such π's), then π fixes precisely the graphs $G \in X$ in which $\{a, b, c\}$ is either a clique or a stable set, and $\{a, b, c\}$ is either complete or anticomplete to $\{d, e\}$. So, $|X^\pi| = 2^3$.

Example 3.2

Find the number of non-isomorphic graphs on five vertices.

Solution (continued).

- If $\pi = (abcd)$, for pairwise distinct $a, b, c, d \in \{1, \dots, 5\}$ (note: there are 30 such π's), then π fixes precisely the graphs $G \in X$ in which all the following hold:
  - ab, bc, cd, da are either all edges or all non-edges,
  - ac and bd are either both edges or both non-edges,
  - the fifth vertex of G (i.e. the unique vertex in $\{1, \dots, 5\} \setminus \{a, b, c, d\}$) is either complete or anticomplete to $\{a, b, c, d\}$.

  So, $|X^\pi| = 2^3$.

Example 3.2

Find the number of non-isomorphic graphs on five vertices.

Solution (continued).

- If $\pi = (abcd)$, for pairwise distinct $a, b, c, d \in \{1, \dots, 5\}$ (note: there are 30 such π's), then π fixes precisely the graphs $G \in X$ in which all the following hold:
  - ab, bc, cd, da are either all edges or all non-edges,
  - ac and bd are either both edges or both non-edges,
  - the fifth vertex of G (i.e. the unique vertex in $\{1, \dots, 5\} \setminus \{a, b, c, d\}$) is either complete or anticomplete to $\{a, b, c, d\}$.

  So, $|X^\pi| = 2^3$.
- If $\pi = (abcde)$, for pairwise distinct a, b, c, d, e (note: there are 24 such π's), then π fixes precisely the graphs $G \in X$ in which both the following hold:
  - ab, bc, cd, de, ea are either all edges or all non-edges,
  - ac, bd, ce, da, eb are either all edges or all non-edges.

  So, $|X^\pi| = 2^2$.

Example 3.2

Find the number of non-isomorphic graphs on five vertices.

Solution (continued). Now, by Burnside's lemma, we see that the number of orbits of our action is

$$
\begin{aligned}
& |X/\mathrm{Sym}(5)| \\
= \; & \tfrac{1}{|\mathrm{Sym}(5)|} \sum_{\pi \in \mathrm{Sym}(5)} |X^\pi| \\
= \; & \tfrac{1}{5!}\left(2^{10} + 10 \cdot 2^7 + 15 \cdot 2^6 + 20 \cdot 2^4 + 20 \cdot 2^3 + 30 \cdot 2^3 + 24 \cdot 2^2\right) \\
= \; & 34.
\end{aligned}
$$

Example 3.2

Find the number of non-isomorphic graphs on five vertices.

Solution (continued). Now, by Burnside's lemma, we see that the number of orbits of our action is

$$
\begin{aligned}
& |X/\mathrm{Sym}(5)| \\
= \; & \frac{1}{|\mathrm{Sym}(5)|} \sum_{\pi \in \mathrm{Sym}(5)} |X^{\pi}| \\
= \; & \frac{1}{5!}\left(2^{10} + 10 \cdot 2^{7} + 15 \cdot 2^{6} + 20 \cdot 2^{4} + 20 \cdot 2^{3} + 30 \cdot 2^{3} + 24 \cdot 2^{2}\right) \\
= \; & 34.
\end{aligned}
$$

So, there are 34 non-isomorphic graphs on five vertices.

- Group action axioms (group G acting on set X):
  - for all $x \in X$, $1_G \cdot x = x$.
  - for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.
- Set of fixed points of $g \in G$: $X^g := \{x \in X \mid g \cdot x = x\}$;
- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;
- Partition of X into orbits of action a: X/G.

Burnside's lemma

Let a be an action of a finite group G on a finite set X. Then

$$|X/G| \quad = \quad \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

- Let's prove Burnside's lemma!

- Given an action a of a group G on a set X, and given $x, y \in X$, we set $M_a(x, y) := \{g \in G \mid g \cdot x = y\}$.

- Given an action a of a group G on a set X, and given $x, y \in X$, we set $M_a(x, y) := \{g \in G \mid g \cdot x = y\}$.
- Note that $M_a(x, x) = G_x$, and that $M_a(x, y) \neq \emptyset$ iff $y \in G \cdot x$.

- Given an action a of a group G on a set X, and given $x, y \in X$, we set $M_a(x, y) := \{g \in G \mid g \cdot x = y\}$.
- Note that $M_a(x, x) = G_x$, and that $M_a(x, y) \neq \emptyset$ iff $y \in G \cdot x$.

Lemma 2.5

Let a be an action of a finite group G on a finite set X, and let $x \in X$. Then for all $y \in G \cdot x$, we have that $|M_a(x, y)| = |G_x|$.

Proof.

- Given an action a of a group G on a set X, and given $x, y \in X$, we set $M_a(x, y) := \{g \in G \mid g \cdot x = y\}$.
- Note that $M_a(x, x) = G_x$, and that $M_a(x, y) \neq \emptyset$ iff $y \in G \cdot x$.

Lemma 2.5

Let a be an action of a finite group G on a finite set X, and let $x \in X$. Then for all $y \in G \cdot x$, we have that $|M_a(x, y)| = |G_x|$.

Proof. Fix $y \in G \cdot x$, and fix any $g_y \in G$ s.t. $g_y \cdot x = y$.

- Given an action a of a group G on a set X, and given $x, y \in X$, we set $M_a(x, y) := \{g \in G \mid g \cdot x = y\}$.
- Note that $M_a(x, x) = G_x$, and that $M_a(x, y) \neq \emptyset$ iff $y \in G \cdot x$.

Lemma 2.5

Let a be an action of a finite group G on a finite set X, and let $x \in X$. Then for all $y \in G \cdot x$, we have that $|M_a(x, y)| = |G_x|$.

Proof. Fix $y \in G \cdot x$, and fix any $g_y \in G$ s.t. $g_y \cdot x = y$. We now define a function $f : G \to G$ by setting $f(g) = g_y g$; since G is a group, f is one-to-one.

- Given an action a of a group G on a set X, and given $x, y \in X$, we set $M_a(x, y) := \{g \in G \mid g \cdot x = y\}$.
- Note that $M_a(x, x) = G_x$, and that $M_a(x, y) \neq \emptyset$ iff $y \in G \cdot x$.

Lemma 2.5

Let a be an action of a finite group G on a finite set X, and let $x \in X$. Then for all $y \in G \cdot x$, we have that $|M_a(x, y)| = |G_x|$.

Proof. Fix $y \in G \cdot x$, and fix any $g_y \in G$ s.t. $g_y \cdot x = y$. We now define a function $f : G \to G$ by setting $f(g) = g_y g$; since G is a group, f is one-to-one. WTS $f[G_x] = M_a(x, y)$; since f is one-to-one, this will imply that $|M_a(x, y)| = |G_x|$, which is what we need.

- $M_a(x, y) := \{g \in G \mid g \cdot x = y\}$

Lemma 2.5

Let a be an action of a finite group G on a finite set X, and let $x \in X$. Then for all $y \in G \cdot x$, we have that $|M_a(x, y)| = |G_x|$.

Proof (continued). Reminder: $y \in G \cdot x$; $g_y \cdot x = y$; $f(g) = g_y g$. WTS $f[G_x] = M_a(x, y)$.

- $M_a(x, y) := \{g \in G \mid g \cdot x = y\}$

Lemma 2.5

Let a be an action of a finite group G on a finite set X, and let $x \in X$. Then for all $y \in G \cdot x$, we have that $|M_a(x, y)| = |G_x|$.

Proof (continued). Reminder: $y \in G \cdot x$; $g_y \cdot x = y$; $f(g) = g_y g$. WTS $f[G_x] = M_a(x, y)$.

First, fix $g \in G_x$. Then $f(g) \cdot x = (g_y g) \cdot x = g_y \cdot (g \cdot x) = g_y \cdot x = y$, and so $f(g) \in M_a(x, y)$. Thus, $f[G_x] \subseteq M_a(x, y)$.

- $M_a(x, y) := \{g \in G \mid g \cdot x = y\}$

Lemma 2.5

Let a be an action of a finite group G on a finite set X, and let $x \in X$. Then for all $y \in G \cdot x$, we have that $|M_a(x, y)| = |G_x|$.

Proof (continued). Reminder: $y \in G \cdot x$; $g_y \cdot x = y$; $f(g) = g_y g$. WTS $f[G_x] = M_a(x, y)$.

First, fix $g \in G_x$. Then
$f(g) \cdot x = (g_y g) \cdot x = g_y \cdot (g \cdot x) = g_y \cdot x = y$, and so
$f(g) \in M_a(x, y)$. Thus, $f[G_x] \subseteq M_a(x, y)$.

On the other hand, fix any $g' \in M_a(x, y)$. Then $g' \cdot x = y$, and so
$(g_y^{-1} g') \cdot x = g_y^{-1} \cdot (g' \cdot x) = g_y^{-1} \cdot y = g_y^{-1} \cdot (g_y \cdot x) =$
$(g_y^{-1} g_y) \cdot x = 1_G \cdot x = x$; consequently, $g_y^{-1} g' \in G_x$. But
$f(g_y^{-1} g') = g_y (g_y^{-1} g') = (g_y g_y^{-1}) g' = 1_G g' = g'$, and so
$M_a(x, y) \subseteq f[G_x]$.

- $M_a(x, y) := \{g \in G \mid g \cdot x = y\}$

Lemma 2.5

Let a be an action of a finite group G on a finite set X, and let $x \in X$. Then for all $y \in G \cdot x$, we have that $|M_a(x, y)| = |G_x|$.

Proof (continued). Reminder: $y \in G \cdot x$; $g_y \cdot x = y$; $f(g) = g_y g$. WTS $f[G_x] = M_a(x, y)$.

First, fix $g \in G_x$. Then $f(g) \cdot x = (g_y g) \cdot x = g_y \cdot (g \cdot x) = g_y \cdot x = y$, and so $f(g) \in M_a(x, y)$. Thus, $f[G_x] \subseteq M_a(x, y)$.

On the other hand, fix any $g' \in M_a(x, y)$. Then $g' \cdot x = y$, and so $(g_y^{-1} g') \cdot x = g_y^{-1} \cdot (g' \cdot x) = g_y^{-1} \cdot y = g_y^{-1} \cdot (g_y \cdot x) = (g_y^{-1} g_y) \cdot x = 1_G \cdot x = x$; consequently, $g_y^{-1} g' \in G_x$. But $f(g_y^{-1} g') = g_y (g_y^{-1} g') = (g_y g_y^{-1}) g' = 1_G g' = g'$, and so $M_a(x, y) \subseteq f[G_x]$.

We have now shown that $f[G_x] = M_a(x, y)$, and we are done.

- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;

The orbit-stabilizer theorem

Let a be an action of a finite group G on a finite set X. Then for all $x \in X$, we have that $|G \cdot x| = \frac{|G|}{|G_x|}$.

Proof.

- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;

The orbit-stabilizer theorem

Let a be an action of a finite group G on a finite set X. Then for all $x \in X$, we have that $|G \cdot x| = \frac{|G|}{|G_x|}$.

Proof. Fix $x \in X$, and note that sets of the form $M_a(x, y)$, with $y \in G \cdot x$, form a partition of G, and so

$$\begin{aligned} |G| &= |\textstyle\bigcup_{y \in G \cdot x} M_a(x, y)| \\ &= \textstyle\sum_{y \in G \cdot x} |M_a(x, y)| \\ &= \textstyle\sum_{y \in G \cdot x} |G_x| && \text{by Lemma 2.5} \\ &= |G \cdot x||G_x|, \end{aligned}$$

and the result follows.

- Set of fixed points of $g \in G$: $X^g := \{x \in X \mid g \cdot x = x\}$;
- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;
- Partition of X into orbits of action a: X/G.

Lemma 2.6

Let a be an action of a finite group G on a finite set X. Then

$$|X/G| = \sum_{x \in X} \frac{1}{|G \cdot x|}.$$

Proof.

- Set of fixed points of $g \in G$: $X^g := \{x \in X \mid g \cdot x = x\}$;
- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;
- Partition of X into orbits of action a: X/G.

Lemma 2.6

Let a be an action of a finite group G on a finite set X. Then

$$|X/G| \;=\; \sum_{x \in X} \frac{1}{|G \cdot x|}.$$

Proof. Set $t := |X/G|$, and let $O_1, \ldots, O_t$ be the orbits of the action a.

- Set of fixed points of $g \in G$: $X^g := \{x \in X \mid g \cdot x = x\}$;
- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;
- Partition of X into orbits of action a: X/G.

Lemma 2.6

Let a be an action of a finite group G on a finite set X. Then

$$|X/G| \;=\; \textstyle\sum_{x \in X} \frac{1}{|G \cdot x|}.$$

Proof. Set $t := |X/G|$, and let $O_1, \ldots, O_t$ be the orbits of the action a. By Proposition 2.4, we have that

- $(O_1, \ldots, O_t)$ is a partition of X;
- for all $i \in \{1, \ldots, t\}$ and $x \in O_i$, $G \cdot x = O_i$.

- Set of fixed points of $g \in G$: $X^g := \{x \in X \mid g \cdot x = x\}$;
- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;
- Partition of X into orbits of action a: X/G.

Lemma 2.6

Let a be an action of a finite group G on a finite set X. Then

$$|X/G| \; = \; \textstyle\sum_{x \in X} \frac{1}{|G \cdot x|}.$$

Proof. Set $t := |X/G|$, and let $O_1, \ldots, O_t$ be the orbits of the action a. By Proposition 2.4, we have that

- $(O_1, \ldots, O_t)$ is a partition of X;
- for all $i \in \{1, \ldots, t\}$ and $x \in O_i$, $G \cdot x = O_i$.

We now compute

$$\textstyle\sum_{x \in X} \frac{1}{|G \cdot x|} \; = \; \sum_{i=1}^{t} \sum_{x \in O_i} \frac{1}{|G \cdot x|} \; = \; \sum_{i=1}^{t} \sum_{x \in O_i} \frac{1}{|O_i|} \; = \; t,$$

which is what we needed.

- Set of fixed points of $g \in G$: $X^g := \{x \in X \mid g \cdot x = x\}$;
- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;
- Partition of X into orbits of action a: X/G.

Burnside's lemma

Let a be an action of a finite group G on a finite set X. Then

$$|X/G| \quad = \quad \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Proof.

- Set of fixed points of $g \in G$: $X^g := \{x \in X \mid g \cdot x = x\}$;
- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;
- Partition of X into orbits of action a: X/G.

Burnside's lemma

Let a be an action of a finite group G on a finite set X. Then

$$|X/G| \;=\; \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Proof. Let $F := \{(g, x) \in G \times X \mid g \cdot x = x\}$. We will count $|F|$ in two ways.

- Set of fixed points of $g \in G$: $X^g := \{x \in X \mid g \cdot x = x\}$;
- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;
- Partition of X into orbits of action a: X/G.

Burnside's lemma

Let a be an action of a finite group G on a finite set X. Then

$$|X/G| \quad = \quad \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Proof. Let $F := \{(g, x) \in G \times X \mid g \cdot x = x\}$. We will count $|F|$ in two ways.

On the one hand, for all $g \in G$ and $x \in X$, we have that $(g, x) \in F$ iff $x \in X^g$; so, $|F| = \sum_{g \in G} |X^g|$.

- Set of fixed points of $g \in G$: $X^g := \{x \in X \mid g \cdot x = x\}$;
- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;
- Partition of X into orbits of action a: X/G.

Burnside's lemma

Let a be an action of a finite group G on a finite set X. Then

$$|X/G| \quad = \quad \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Proof (continued). Reminder: $F := \{(g, x) \in G \times X \mid g \cdot x = x\}$; $|F| = \sum_{g \in G} |X^g|$.

- Set of fixed points of $g \in G$: $X^g := \{x \in X \mid g \cdot x = x\}$;
- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;
- Partition of X into orbits of action a: X/G.

Burnside's lemma

Let a be an action of a finite group G on a finite set X. Then

$$|X/G| \quad = \quad \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Proof (continued). Reminder: $F := \{(g, x) \in G \times X \mid g \cdot x = x\}$;
$|F| = \sum_{g \in G} |X^g|$.

On the other hand, for all $g \in G$ and $x \in X$, we have that $(g, x) \in F$ iff $g \in G_x$, and so

$$\begin{aligned} |F| &= \sum_{x \in X} |G_x| \\ &= \sum_{x \in X} \frac{|G|}{|G \cdot x|} && \text{by the orbit-stabilizer theorem} \\ &= |G| \sum_{x \in X} \frac{1}{|G \cdot x|}. \end{aligned}$$

- Set of fixed points of $g \in G$: $X^g := \{x \in X \mid g \cdot x = x\}$;
- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;
- Partition of X into orbits of action a: X/G.

Burnside's lemma

Let a be an action of a finite group G on a finite set X. Then

$$|X/G| \;=\; \tfrac{1}{|G|} \textstyle\sum_{g \in G} |X^g|.$$

Proof (continued). Reminder: $F := \{(g, x) \in G \times X \mid g \cdot x = x\}$;
$|F| = \sum_{g \in G} |X^g|$; $|F| = |G| \sum_{x \in X} \frac{1}{|G \cdot x|}$.

- Set of fixed points of $g \in G$: $X^g := \{x \in X \mid g \cdot x = x\}$;
- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;
- Partition of X into orbits of action a: X/G.

Burnside's lemma

Let a be an action of a finite group G on a finite set X. Then

$$|X/G| \;=\; \tfrac{1}{|G|} \sum_{g \in G} |X^g|.$$

Proof (continued). Reminder: $F := \{(g, x) \in G \times X \mid g \cdot x = x\}$; $|F| = \sum_{g \in G} |X^g|$; $|F| = |G| \sum_{x \in X} \frac{1}{|G \cdot x|}$.

So, $|G| \sum_{x \in X} \frac{1}{|G \cdot x|} = \sum_{g \in G} |X^g|$,

- Set of fixed points of $g \in G$: $X^g := \{x \in X \mid g \cdot x = x\}$;
- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;
- Partition of X into orbits of action a: X/G.

Burnside's lemma

Let a be an action of a finite group G on a finite set X. Then

$$|X/G| \quad = \quad \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Proof (continued). Reminder: $F := \{(g, x) \in G \times X \mid g \cdot x = x\}$;
$|F| = \sum_{g \in G} |X^g|$; $|F| = |G| \sum_{x \in X} \frac{1}{|G \cdot x|}$.

So, $|G| \sum_{x \in X} \frac{1}{|G \cdot x|} = \sum_{g \in G} |X^g|$, and consequently,

$$\sum_{x \in X} \frac{1}{|G \cdot x|} \quad = \quad \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

- Set of fixed points of $g \in G$: $X^g := \{x \in X \mid g \cdot x = x\}$;
- Stabilizer of $x \in X$: $G_x := \{g \in G \mid g \cdot x = x\}$;
- Orbit of $x \in X$: $G \cdot x := \{g \cdot x \mid g \in G\}$;
- Partition of X into orbits of action a: X/G.

Burnside's lemma

Let a be an action of a finite group G on a finite set X. Then

$$|X/G| \quad = \quad \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Proof (continued). Reminder: $F := \{(g, x) \in G \times X \mid g \cdot x = x\}$; $|F| = \sum_{g \in G} |X^g|$; $|F| = |G| \sum_{x \in X} \frac{1}{|G \cdot x|}$.

So, $|G| \sum_{x \in X} \frac{1}{|G \cdot x|} = \sum_{g \in G} |X^g|$, and consequently,

$$\sum_{x \in X} \frac{1}{|G \cdot x|} \quad = \quad \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

But by Lemma 2.6, $|X/G| = \sum_{x \in X} \frac{1}{|G \cdot x|}$, and the result follows.