NDMI012: Combinatorics and Graph Theory 2

Lecture #10

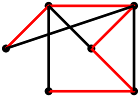
Hamiltonian graphs

Irena Penev

April 26, 2022

Definition

A *Hamiltonian path* (or a *Hamilton path*) of a graph G is a path of G that passes through all vertices of G.



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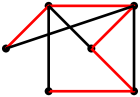

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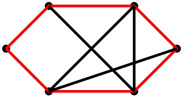
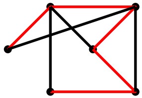
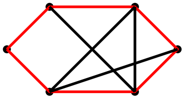
A *Hamiltonian cycle* (or a *Hamilton cycle*) of a graph G is a cycle of G that passes through all vertices of G.



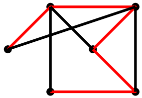


Hamiltonian
path



Hamiltonian
cycle

- Obviously, every graph that contains a Hamiltonian cycle also contains a Hamiltonian path.



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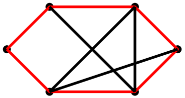

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- Obviously, every graph that contains a Hamiltonian cycle also contains a Hamiltonian path.

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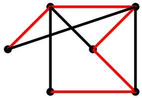
A graph is *Hamiltonian* if it has a Hamiltonian cycle.



Hamiltonian
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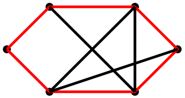

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A graph is *Hamiltonian* if it has a Hamiltonian cycle.

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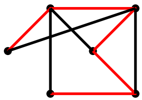

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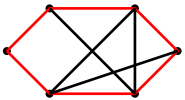

Hamiltonian
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- Obviously, every graph that contains a Hamiltonian cycle also contains a Hamiltonian path.

Definition

A graph is *Hamiltonian* if it has a Hamiltonian cycle.

- It is NP-hard to determine if a graph is Hamiltonian.
- Nevertheless, there are a number sufficient conditions for Hamiltonicity, which can easily be checked in polynomial time. (We will state and prove a few such conditions.)

Definition

For a real number $t > 0$, a graph G is *t-tough* if $\forall S \subsetneq V(G)$, the graph $G \setminus S$ has at most $\max\{1, \frac{|S|}{t}\}$ many components.

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Conjecture [Chvátal]

There exists some $t > 0$ s.t. every t-tough graph is Hamiltonian.

- Chvátal's conjecture remains open.

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Conjecture [Chvátal]

There exists some $t > 0$ s.t. every t-tough graph is Hamiltonian.

- Chvátal's conjecture remains open.
- However, we have the following easy proposition.

Proposition 1.2

Every Hamiltonian graph is 1-tough.

Definition

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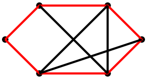

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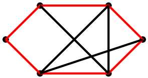

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Every Hamiltonian graph is 1-tough.

Proof. Let G be a Hamiltonian graph, and let $S \subsetneq V(G)$. Since G is Hamiltonian, it is connected; so, if $S = \emptyset$, then $G \setminus S = G$ has only one component, and we are done.

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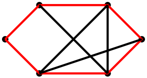

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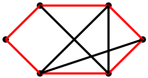

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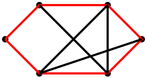

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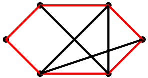

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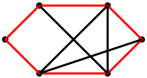

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Lemma 2.1

Let G be a graph, and let x and y be distinct, non-adjacent vertices of G that satisfy $d_G(x) + d_G(y) \geq |V(G)|$. Then G is Hamiltonian iff $G + xy$ is Hamiltonian.

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Proof (continued). Let $S_x := \{i \mid 1 \leq i \leq n-1,\ xc_{i+1} \in E(G)\}$ and $S_y := \{i \mid 1 \leq i \leq n-1,\ yc_i \in E(G)\}$.

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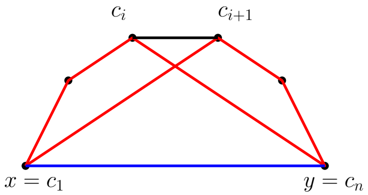

But now $\underbrace{x}_{=c_1}, c_2, \dots, c_i, \underbrace{y}_{=c_n}, c_{n-1}, \dots, c_{i+1}, \underbrace{x}_{=c_1}$ is a Hamiltonian cycle of G, and so G is Hamiltonian.

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Definition

The *Chvátal closure* of a graph G is the graph obtained by repeatedly adding edges between non-adjacent vertices x, y s.t. $d(x) + d(y) \geq |V(G)|$, until no more such edges can be added.

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- The Chvátal closure of a graph is uniquely defined (i.e. the order in which edges are added does not matter).

Theorem 2.2

A graph is Hamiltonian iff its Chvátal closure is Hamiltonian.

Proof. This follows from Lemma 2.1 by an easy induction.

Theorem 2.3 [Ore]

Let G be a graph on at least three vertices. Assume that for all distinct, non-adjacent vertices x, y of G, we have that $d_G(x) + d_G(y) \geq |V(G)|$. Then G is Hamiltonian.

Proof. The Chvátal closure of G is the complete graph on $|V(G)|$ vertices, which (since $|V(G)| \geq 3$) is clearly Hamiltonian. So, by Theorem 2.2, G is also Hamiltonian.

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Theorem 2.4 [Dirac]

Let G be a graph on at least three vertices. If $\delta(G) \geq \frac{|V(G)|}{2}$, then G is Hamiltonian.

Proof. This is an immediate corollary of Theorem 2.3.

Definition

Let $\mathbf{a} = (a_1, \ldots, a_n)$ be a list (vector) of integers s.t. $0 \leq a_1 \leq \cdots \leq a_n \leq n-1$. A graph G on n vertices *dominates* $\mathbf{a}$ if for some ordering $v_1, \ldots, v_n$ of the vertices of G, we have that $d_G(v_1) \geq a_1, \ldots, d_G(v_n) \geq a_n$. We say that $\mathbf{a}$ is *Hamiltonian* if every n-vertex graph that dominates $\mathbf{a}$ is Hamiltonian.

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Theorem 2.5

Let $n \geq 3$ be an integer, and let $\mathbf{a} = (a_1, \ldots, a_n)$ be a sequence of integers s.t. $0 \leq a_1 \leq \cdots \leq a_n \leq n-1$. Then the following are equivalent:

(a) for all indices $i < \frac{n}{2}$, if $a_i \leq i$, then $a_{n-i} \geq n-i$;

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Proof. Suppose first that (a) holds; we must prove (b). Suppose otherwise.

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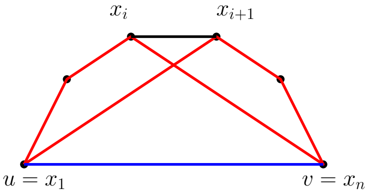
Proof (continued). Let $S := \{i \mid 1 \leq i \leq n-1,\ ux_{i+1} \in E(G)\}$; clearly, $s := |S| = d_G(u)$. If there exists some $i \in S$ s.t. $vx_i \in E(G)$, then $\underbrace{x_1}_{=u}, x_2, \ldots, x_i, \underbrace{x_n}_{=v}, x_{n-1}, \ldots, x_{i+1}, \underbrace{x_1}_{=u}$ would be a Hamiltonian cycle in G, contrary to the fact that G is not Hamiltonian.



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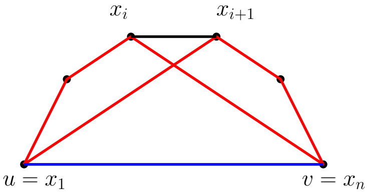
Proof (continued). Let $S := \{i \mid 1 \leq i \leq n-1,\ ux_{i+1} \in E(G)\}$; clearly, $s := |S| = d_G(u)$. If there exists some $i \in S$ s.t. $vx_i \in E(G)$, then $\underbrace{x_1}_{=u}, x_2, \ldots, x_i, \underbrace{x_n}_{=v}, x_{n-1}, \ldots, x_{i+1}, \underbrace{x_1}_{=u}$ would be a Hamiltonian cycle in G, contrary to the fact that G is not Hamiltonian.



So, no such i exists, and it follows that $d_G(v) \leq n - 1 - s$.

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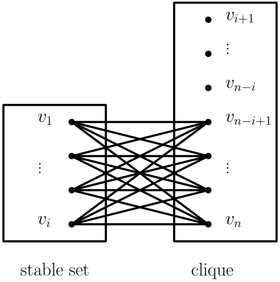

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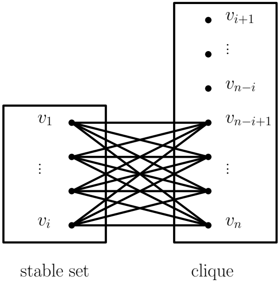

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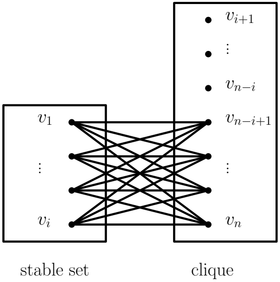

Also, the graph is not 1-tough, because deleting $\{v_{n-i+1}, \ldots, v_n\}$ yields a graph with $i + 1$ components. So, by Proposition 1.2, G is not Hamiltonian, and it follows that (b) does not hold.

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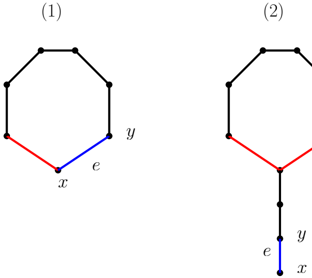
Proof. Let $e = xy$ be an edge of G; we must show that e belongs to an even number of Hamiltonian cycles of G.

A *lollipop* is a connected subgraph H of G s.t. $V(H) = V(G)$, $e \in E(H)$, and H satisfies one of the following:

(1) H is a cycle;

(2) $d_H(x) = 1$, H has one vertex of degree three, and all other vertices of H are of degree two.

Proof (continued).



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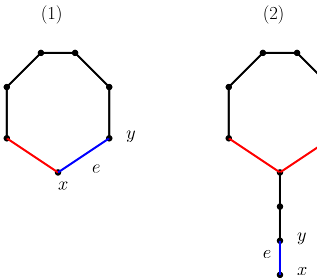

If H is a lollipop that satisfies (1), then H has a unique *tail*, namely the unique edge of H incident with x and distinct from e. On the other hand, if H is a lollipop that satisfies (2), then H has two *tails*, namely, the two edges of the unique cycle of H that are incident with the unique vertex of degree three in H.

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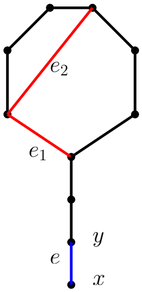
For example, in the picture below, if H_i (for $i \in \{1, 2\}$) consists of the blue and black edges, plus the red edge e_i, then lollipops H_1 and H_2 are adjacent in L.



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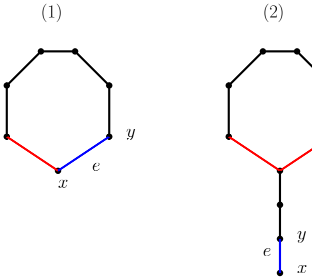

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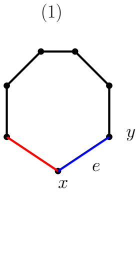
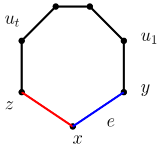


This is enough because the number of odd-degree vertices in L is even (true for any graph), and the lollipops satisfying (1) are precisely the Hamiltonian cycles of G that contain the edge e.

Proof (continued). Suppose that $H = x, y, u_1, \ldots, u_t, z, x$ $(t \geq 0)$ is a lollipop satisfying (1), i.e. H is a Hamiltonian cycle of G containing e.



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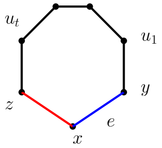

Then xz is the unique tail of H, and the neighbors of H in L are precisely the graphs that can be obtained from $H - xz$ by adding an edge between z and a vertex in $N_G(z) \setminus N_H(z)$.

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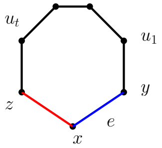
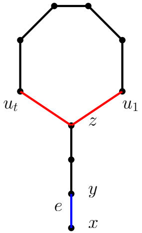
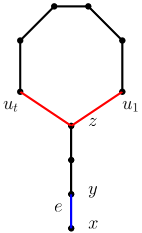


Then xz is the unique tail of H, and the neighbors of H in L are precisely the graphs that can be obtained from $H - xz$ by adding an edge between z and a vertex in $N_G(z) \setminus N_H(z)$. So, $d_L(H) = |N_G(z) \setminus N_H(z)| = d_G(z) - 2$; since $d_G(z)$ is odd, so is $d_L(H)$.

Proof (continued). Suppose now that H is a lollipop satisfying (2); let $z, u_1, \ldots, u_t, z$ ($t \geq 2$) be the unique cycle of H, where z is the unique vertex of degree three in H.



Proof (continued).



Then the lollipop H has two tails, namely zu_1 and zu_t, and the neighbors of H in L are precisely the graphs that an be obtained in one of the following two ways as follows:

- by starting with $H - zu_1$, and then adding an edge between u_1 and $N_G(u_1) \setminus \{z, u_2\}$;
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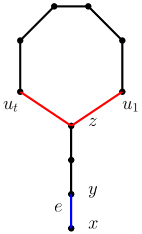

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- by starting with $H - zu_1$, and then adding an edge between u_1 and $N_G(u_1) \setminus \{z, u_2\}$;
- by starting with $H - zu_t$, and then adding an edge between u_t and $N_G(u_t) \setminus \{z, u_{t-1}\}$.

So, $d_L(H) = (d_G(u_1) - 2) + (d_G(u_t) - 2) = d_G(u_1) + d_G(u_t) - 4$. Since all vertices of G have odd degree, we deduce that $d_L(H)$ is even.

Lemma 3.1

Let G be a graph in which all vertices are of odd degree. Then every edge of G belongs to an even number of Hamiltonian cycles. In particular, every edge of G that belongs to a Hamiltonian cycle, belongs to at least two Hamiltonian cycles.

Proof (continued). We have now shown that the odd-degree vertices of our auxiliary graph L are precisely the Hamiltonian cycles of H that contain the edge e. This completes the argument.