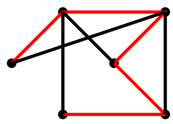
NDMI012: Combinatorics and Graph Theory 2

Lecture #10
Hamiltonian graphs

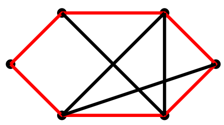
Irena Penev

1 Hamiltonian graphs and t-toughness

A *Hamiltonian path* (or a *Hamilton path*) of a graph G is a path of G that passes through all vertices of G. An example is shown below (the Hamiltonian path is in red.)



A *Hamiltonian cycle* (or a *Hamilton cycle*) of a graph G is a cycle of G that passes through all vertices of G. An example is shown below (the Hamiltonian cycle is in red.)



A graph is *Hamiltonian* if it has a Hamiltonian cycle.

We remark that it is NP-hard to determine whether a graph is Hamiltonian. This is in contrast with Eulerian graphs: to check if a graph is Eulerian, we need only check if it is connected and all vertices are of even degree, which can obviously be done in polynomial time. Nevertheless, there are a number sufficient conditions for Hamiltonicity, which can easily be checked in polynomial time (see section 2 below).

For a real number $t > 0$, a graph G is *t-tough* if for every set $S \subsetneqq V(G)$, the graph $G \setminus S$ has at most $\max\{1, \frac{|S|}{t}\}$ many components.[1]

[1] Equivalently, for a real number $t > 0$, a graph G is *t-tough* if for every set $S \subsetneqq V(G)$, the graph $G \setminus S$ either is connected or has at most $\frac{|S|}{t}$ many components.

Conjecture 1.1 (Chvátal)**.** *There exists some $t > 0$ such that every t-tough graph is Hamiltonian.*

The conjecture above remains open. We do have the following simple proposition, though.

Proposition 1.2. *Every Hamiltonian graph is 1-tough.*

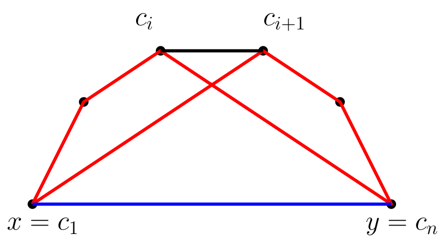
Proof. Let G be a Hamiltonian graph, and let $S \subsetneq V(G)$. Since G is Hamiltonian, it is connected; so, if $S = \emptyset$, then $G \setminus S = G$ has only one component, and we are done. We may now assume that $S \neq \emptyset$. Let C be a Hamiltonian cycle in G. Clearly, $C \setminus S$ is the disjoint union of at most $|S|$ many paths, and so $C \setminus S$ has at most $|S|$ many components. Since C is a spanning subgraph of G,[2] it is clear that $G \setminus S$ has no more components than $C \setminus S$ does.[3] So, $G \setminus S$ has at most $|S|$ many components, and the result follows. $\square$

2 Hamiltonian graphs and vertex degrees

Lemma 2.1. *Let G be a graph, and let x and y be distinct, non-adjacent vertices of G that satisfy $d_G(x) + d_G(y) \geq |V(G)|$. Then G is Hamiltonian if and only if $G + xy$ is Hamiltonian.*

Proof. It is clear that if G is Hamiltonian, then so is $G + xy$.[4]

Suppose now that $G + xy$ is Hamiltonian; we must show that G is Hamiltonian. Let C be a Hamiltonian cycle of $G + xy$. If $xy \notin E(C)$, then C is a Hamiltonian cycle of G, and we are done. So, assume that $xy \in E(C)$. Now, consider the path $C - xy = c_1, \dots, c_n$, with $c_1 = x$ and $c_n = y$.[5] Let $S_x := \{i \mid 1 \leq i \leq n-1,\ xc_{i+1} \in E(G)\}$ and $S_y := \{i \mid 1 \leq i \leq n-1,\ yc_i \in E(G)\}$. Note that $|S_x| + |S_y| = d_G(x) + d_G(y) \geq |V(G)|$, whereas $|S_x \cup S_y| \leq |V(G)| - 1$. So, $S_x \cap S_y \neq \emptyset$. Fix $i \in S_x \cap S_y$. Since $x = c_1$ and $y = c_n$ are non-adjacent in G, we see that $2 \leq i \leq n-2$. But now $\underbrace{x}_{=c_1}, c_2, \dots, c_i, \underbrace{y}_{=c_n}, c_{n-1}, \dots, c_{i+1}, \underbrace{x}_{=c_1}$ is a Hamiltonian cycle of G, and so G is Hamiltonian.



[2]A *spanning subgraph* of a graph G is a subgraph of G that contains all vertices of G.

[3]Indeed, $G \setminus S$ can be obtained from $C \setminus S$ by possibly adding edges, and adding edges cannot increase the number of components.

[4]Indeed, any Hamiltonian cycle of G is also a Hamiltonian cycle of $G + xy$.

[5]Since C is a Hamiltonian cycle of $G + xy$, we have that $V(G) = V(C) = \{c_1, \dots, c_n\}$.

□

The *Chvátal closure* of a graph G is the graph obtained by repeatedly adding edges between non-adjacent vertices x, y such that $d(x) + d(y) \geq |V(G)|$, until no more such edges can be added. It is easy to see that the Chvátal closure of a graph is uniquely defined (i.e. the order in which edges are added does not matter).[6]

Theorem 2.2. *A graph is Hamiltonian if and only if its Chvátal closure is Hamiltonian.*

Proof. This follows from Lemma 2.1 by an easy induction. □

Theorem 2.3 (Ore)**.** *Let G be a graph on at least three vertices. Assume that for all distinct, non-adjacent vertices x, y of G, we have that $d_G(x)+d_G(y) \geq |V(G)|$. Then G is Hamiltonian.*

Proof. The Chvátal closure of G is the complete graph on $|V(G)|$ vertices, which (since $|V(G)| \geq 3$) is clearly Hamiltonian. So, by Theorem 2.2, G is also Hamiltonian. □

Theorem 2.4 (Dirac)**.** *Let G be a graph on at least three vertices. If $\delta(G) \geq \frac{|V(G)|}{2}$, then G is Hamiltonian.*

Proof. This is an immediate corollary of Theorem 2.3. □

Let $\mathbf{a} = (a_1, \dots, a_n)$ be a list (vector) of integers such that $0 \leq a_1 \leq \dots \leq a_n \leq n-1$. A graph G on n vertices *dominates* $\mathbf{a}$ if for some ordering $v_1, \dots, v_n$ of the vertices of G, we have that $d_G(v_1) \geq a_1, \dots, d_G(v_n) \geq a_n$. We say that $\mathbf{a}$ is *Hamiltonian* if every n-vertex graph that dominates $\mathbf{a}$ is Hamiltonian.

Theorem 2.5. *Let $n \geq 3$ be an integer, and let $\mathbf{a} = (a_1, \dots, a_n)$ be a sequence of integers such that $0 \leq a_1 \leq \dots \leq a_n \leq n-1$. Then the following are equivalent:*

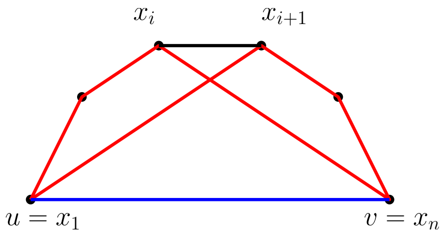
(a) for all indices $i < \frac{n}{2}$, if $a_i \leq i$, then $a_{n-i} \geq n-i$;

(b) $\mathbf{a}$ is Hamiltonian.

Proof. Suppose first that (a) holds; we must prove (b). Suppose otherwise. Then there exists a graph on n vertices that dominates $\mathbf{a}$, but is not Hamiltonian; among all such graphs, let G be one with as many edges as possible. Since G has at least three vertices and is not Hamiltonian, we see that G is not complete. Fix distinct, non-adjacent vertices $u, v \in V(G)$ such that $d_G(u)+d_G(v)$ is maximum; by symmetry, we may assume that $d_G(u) \leq d_G(v)$.

[6] Details?

Then $G+uv$ dominates $\mathbf{a}$ and has more edges than G, and so $G+uv$ is Hamiltonian. Let C be a Hamiltonian cycle in $G+uv$. Then $uv \in E(C)$, for otherwise, C would be a Hamiltonian cycle in G, contrary to the fact that G is not Hamiltonian. We now consider the path $C-uv = x_1,\dots,x_n$, with $x_1 = u$ and $x_n = v$. Let $S := \{i \mid 1 \le i \le n-1,\ ux_{i+1} \in E(G)\}$; clearly, $s := |S| = d_G(u)$. If there exists some $i \in S$ such that $vx_i \in E(G)$, then $\underbrace{x_1}_{=u}, x_2,\dots,x_i, \underbrace{x_n}_{=v}, x_{n-1},\dots,x_{i+1}, \underbrace{x_1}_{=u}$ would be a Hamiltonian cycle in G, contrary to the fact that G is not Hamiltonian.



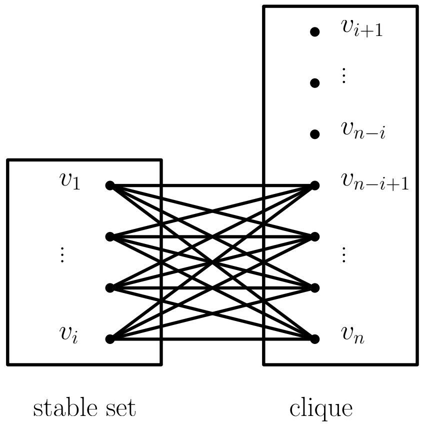
So, no such i exists, and it follows that $d_G(v) \le n-1-s$. But now $d_G(u)+d_G(v) \le s+(n-1-s) = n-1$; since $d_G(u) \le d_G(v)$, we deduce that $d_G(u) < \frac{n}{2}$, and so $s < \frac{n}{2}$. Further, by the maximality of $d_G(u)+d_G(v)$, we see that for all $i \in S$, we have that $d_G(x_i) \le d_G(u) = s$.[7] So, at least s vertices of G (i.e. all the x_i's with $i \in S$) have degree at most $s < \frac{n}{2}$ in G, and it follows that $a_1,\dots,a_s \le s < \frac{n}{2}$.[8] But since $a_s \le s < \frac{n}{2}$, (a) guarantees that $a_{n-s} \ge n-s$; but now $n-s \le a_{n-s} \le \dots \le a_n$, i.e. at least $s+1$ vertices of G have degree at least $n-s$. Since $d_G(u) = s$, we see that u is non-adjacent to at least one of these $s+1$ vertices, call it y. But now $d_G(u)+d_G(y) \ge s+(n-s) = n > n-1 \ge d_G(u)+d_G(v)$, contrary to the maximality of $d_G(u)+d_G(v)$. So, (b) holds.

Suppose now that (a) does not hold; we must show that (b) does not hold either.[9] Since (a) does not hold, there exists some index $i < \frac{n}{2}$ such that $a_i \le i$ and $a_{n-i} \le n-i-1$. Let G be the graph with vertex set $\{v_1,\dots,v_n\}$, with adjacency as follows:

- $\{v_{i+1},\dots,v_n\}$ is a clique;
- $\{v_1,\dots,v_i\}$ is complete to $\{v_{n-i+1},\dots,v_n\}$;
- there are no other edges in G.

The graph G is represented below.

[7] Here, we are using the fact that v is non-adjacent to all vertices x_i with $i \in S$.
[8] We are using the fact that $a_1 \le \dots \le a_n$, and that G dominates $\mathbf{a}$.
[9] So, we must exhibit an n-vertex graph that dominates $\mathbf{a}$ and is not Hamiltonian.



Then

- $d_G(v_1) = \cdots = d_G(v_i) = i \geq a_i \geq \cdots \geq a_1$;
- $d_G(v_{i+1}) = \cdots = d_G(v_{n-i}) = n - i - 1 \geq a_{n-i} \geq \cdots \geq a_{i+1}$;
- $d_G(v_{n-i+1}) = \cdots = d_G(v_n) = n - 1 \geq a_n \geq \cdots \geq a_{n-i+1}$.

So, G dominates $\mathbf{a}$. On the other hand, $G \setminus \{v_{n-i+1}, \ldots, v_n\}$ has $i + 1$ components, and so G is not 1-tough; so, by Proposition 1.2, G is not Hamiltonian, and it follows that (b) does not hold. □

3 Number of Hamiltonian cycles

Lemma 3.1. *Let G be a graph in which all vertices are of odd degree. Then every edge of G belongs to an even number of Hamiltonian cycles.*[10] *In particular, every edge of G that belongs to a Hamiltonian cycle, belongs to at least two Hamiltonian cycles.*

Proof. Let $e = xy$ be an edge of G; we must show that e belongs to an even number of Hamiltonian cycles of G.

A *lollipop* is a connected subgraph H of G such that $V(H) = V(G)$,[11] $e \in E(H)$, and H satisfies one of the following:

(1) H is a cycle;

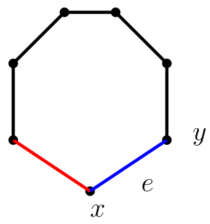
(2) $d_H(x) = 1$, H has one vertex of degree three, and all other vertices of H are of degree two.

[10] It is possible that an edge of G does not belong to any Hamiltonian cycles of G, and indeed, it is possible that G is not Hamiltonian: zero counts as an even number.

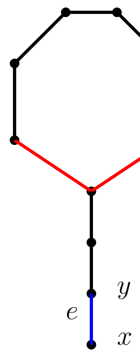
[11] So, H is a spanning subgraph of G.

Note that lollipops satisfying (1) are precisely the Hamiltonian cycles of G that contain the edge e. On the other hand, in case (2), H consists of a cycle, plus a path that has exactly one vertex in common with the cycle, and furthermore, x is the endpoint of this path that does not belong to the cycle. The two types of lollipop are represented below (the edge $e = xy$ is in blue).[12]

(1)



(2)



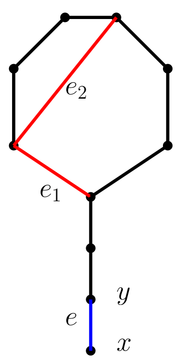
If H is a lollipop that satisfies (1), then H has a unique *tail*, namely the unique edge of H incident with x and distinct from e. On the other hand, if H is a lollipop that satisfies (2), then H has two *tails*, namely, the two edges of the unique cycle of H that are incident with the unique vertex of degree three in H. (In the picture above the tails are in red.)

We now form an auxiliary graph L, as follows. The vertices of L are the lollipops. Two lollipops, H_1 and H_2, are adjacent in L if and only if there exist tails e_1 of H_1 and e_2 of H_2 such that $H_1 - e_1 = H_2 - e_2$.[13]

Our goal is to show that the odd-degree vertices of the auxiliary graph L are precisely the lollipops satisfying (1). This is enough because the number

[12]In case (2), it is possible that y is in fact the unique vertex of H of degree three.

[13]For example, in the picture below, if H_i (for $i \in \{1, 2\}$) consists of the blue and black edges, plus the red edge e_i, then lollipops H_1 and H_2 are adjacent in L.



of odd-degree vertices in L is even,[14] and the lollipops satisfying (1) are precisely the Hamiltonian cycles of G that contain the edge e.

Suppose that $H = x, y, u_1, \dots, u_t, z, x$ ($t \geq 0$) is a lollipop satisfying (1), i.e. H is a Hamiltonian cycle of G containing e. Then xz is the unique tail of H, and the neighbors of H in L are precisely the graphs that can be obtained from $H - xz$ by adding an edge between z and a vertex in $N_G(z) \setminus N_H(z)$. So, $d_L(H) = |N_G(z) \setminus N_H(z)| = d_G(z) - 2$; since $d_G(z)$ is odd, so is $d_L(H)$.

Suppose now that H is a lollipop satisfying (2); let $z, u_1, \dots, u_t, z$ ($t \geq 2$) be the unique cycle of H, where z is the unique vertex of degree three in H. Then the lollipop H has two tails, namely zu_1 and zu_t, and the neighbors of H in L are precisely the graphs that an be obtained in one of the following two ways as follows:

- by starting with $H - zu_1$, and then adding an edge between u_1 and $N_G(u_1) \setminus \{z, u_2\}$;
- by starting with $H - zu_t$, and then adding an edge between u_t and $N_G(u_t) \setminus \{z, u_{t-1}\}$.

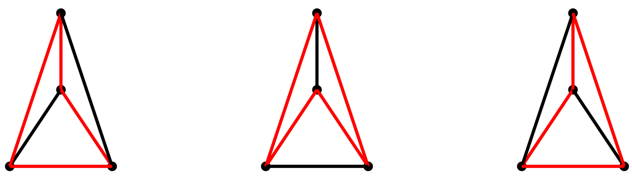
So, $d_L(H) = (d_G(u_1) - 2) + (d_G(u_t) - 2) = d_G(u_1) + d_G(u_t) - 4$. Since all vertices of G have odd degree, we deduce that $d_L(H)$ is even.

We have now shown that the odd-degree vertices of our auxiliary graph L are precisely the Hamiltonian cycles of H that contain the edge e. This completes the argument. □

Theorem 3.2. *Let G be a Hamiltonian graph, all of whose vertices are of odd degree. Then G has at least three Hamiltonian cycles.*

Proof. Let C_1 be a Hamiltonian cycle of G, and let e be some edge of C_1. Then by Lemma 3.1, there exists a Hamiltonian cycle $C_2 \neq C_1$ that also contains the edge e. Since C_1, C_2 are distinct Hamiltonian cycles, we see that there exists an edge $e_1 \in E(C_1) \setminus E(C_2)$; but then Lemma 3.1 guarantees that there exists a Hamiltonian cycle $C_3 \neq C_1$ that contains e_1. Since $e_1 \in E(C_3) \setminus E(C_2)$, we see that $C_3 \neq C_2$. But now C_1, C_2, C_3 are pairwise distinct Hamiltonian cycles of G. □

We note that the bound from Theorem 3.2 is best possible: indeed, K_4 has precisely three Hamiltonian cycles (see the picture below; the Hamiltonian cycles are in red).



[14]Indeed, every graph has an even number of odd-degree vertices. This follows from the fact that the sum of degrees in any graph is even (because it is equal to twice the number of edges).