NDMI012: Combinatorics and Graph Theory 2

Lecture #9

The Tutte polynomial

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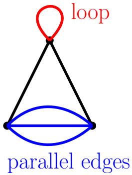
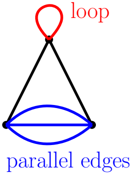
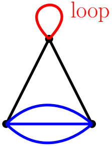
Definition

A *multigraph* is an ordered pair $G = (V(G), E(G))$ s.t. $V(G)$ and $E(G)$ are finite sets (called the *vertex set* and *edge set*, respectively), and each edge (i.e. element of $E(G)$) is associated with two (possibly identical) vertices (i.e. elements of $V(G)$), called its *endpoints*. If an edge has only one endpoint (i.e. its two endpoints are the same), then this edge is called a *loop*. If two distinct edges have the same endpoints, then those edges are *parallel*.





- A *proper (vertex) coloring* of a loopless multigraph G is an assignment of colors to the vertices of G in such a way that, whenever two distinct vertices are joined by an edge (i.e. are the endpoints of the same edge), they receive different colors.



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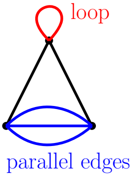

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- If a multigraph has a loop, then it has no proper colorings.
- A *k-coloring* of a multigraph G is a proper coloring of G that uses colors $1, \ldots, k$ (not all of these colors need be used).

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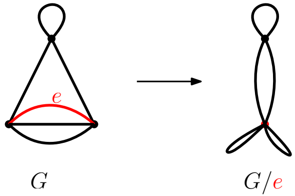
- If e is a non-loop edge of a multigraph G, then the multigraph G/e obtained by *contracting* e is the multigraph obtained by first deleting e, and then identifying its endpoints to a single vertex.



  - Note that edges parallel to e become loops, and it is also possible that new parallel edges are created.

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- There are a number of such polynomials.
- Here, we consider two: the chromatic polynomial and the Tutte polynomial.

Lemma 2.1

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- If G is a loopless multigraph, then $\chi(G)$ is equal to the smallest non-negative integer k s.t. $\pi_G(k) \neq 0$.

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- The *chromatic polynomial* of a multigraph G is the polynomial π_G from the statement of Lemma 2.1.
- If G is a loopless multigraph, then $\chi(G)$ is equal to the smallest non-negative integer k s.t. $\pi_G(k) \neq 0$.
- Note that this implies that computing the chromatic polynomial is NP-hard.
- However, in some special cases, the chromatic polynomial is easy to compute. For example:
  - $\pi_{K_n}(x) = x(x-1)(x-2)\dots(x-n+1)$;
  - $\pi_T(x) = x(x-1)^{n-1}$, for any tree T on n vertices.

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Since π_{G-e} and $\pi_{G/e}$ are of degree at most $|V(G)|$, so is π_G. Now, fix a non-negative integer k. We must show that there are precisely $\pi_G(k)$ many k-colorings of G.

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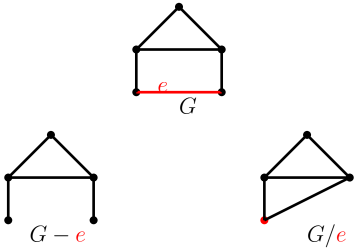

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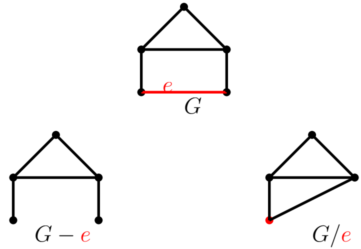

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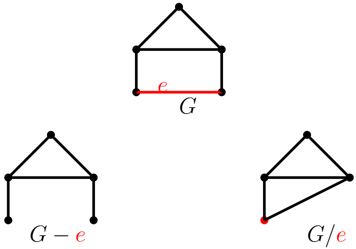

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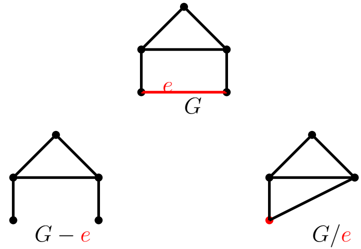

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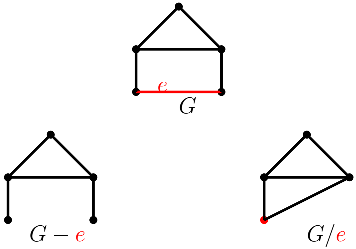

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- Reminder: The polynomial π_G is called the *chromatic polynomial* of G.
- The proof of that lemma in fact gives us a recursive formula for π_G, as follows:
  - if G is edgeless, then $\pi_G(x) = x^{|V(G)|}$;
  - if G has a loop, then $\pi_G(x) = 0$;
  - if G is loopless and has at least one edge, say e, then

$$\pi_G(x) \quad = \quad \pi_{G-e}(x) - \pi_{G/e}(x).$$

  Note that $G - e$ and G/e have fewer edges than G, and so our formula really is recursive.

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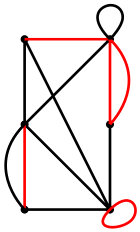

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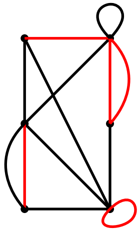

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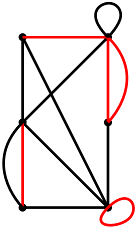

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- In the example above (where the edges of A are in red), we have that $k(G) = 1$, $k_G(A) = 3$, $|A| = 5$, and $|V(G)| = 6$; so, $r_G(A) = 2$ and $c_G(A) = 2$.

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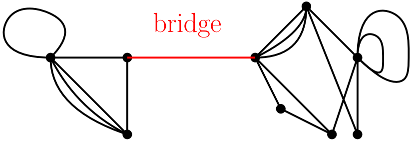
- Since it is NP-hard to compute the chromatic polynomial, it is NP-hard to compute the Tutte polynomial.
- Clearly, if G is edgeless, then $T_G(x, y) = 1$.
- Otherwise, we can get a recursive formula for $T_G(x, y)$, as follows (two slides from now; first, a definition).

Definition

A *bridge* in a multigraph G is an edge e of G s.t. $G - e$ has more components than G.



- Reminder:
  - $r_G(A) := k_G(A) - k(G)$;
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  - $T_G(x,y) := \sum_{A \subseteq E(G)} (x-1)^{r_G(A)}(y-1)^{c_G(A)}$.

Lemma 3.1

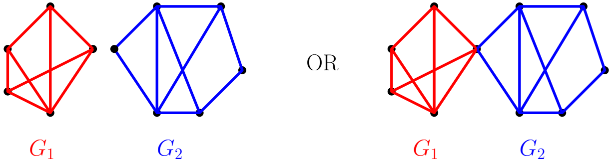
Let e be an edge of a multigraph G. Then

$$T_G(x,y) = \begin{cases} xT_{G/e}(x,y) & \text{if } e \text{ is a bridge of } G \\ yT_{G-e}(x,y) & \text{if } e \text{ is a loop of } G \\ T_{G-e}(x,y) + T_{G/e}(x,y) & \text{otherwise} \end{cases}$$

Proof. Lecture Notes.

Lemma 3.2

If multigraphs G_1 and G_2 have at most one vertex and no edges in common, then $T_{G_1 \cup G_2} = T_{G_1} T_{G_2}$.



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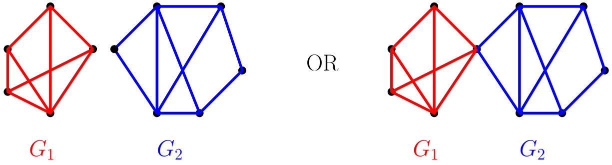

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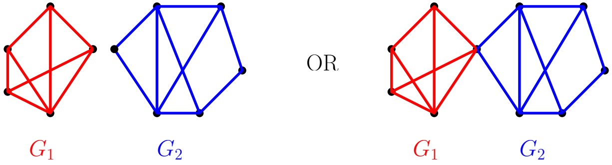

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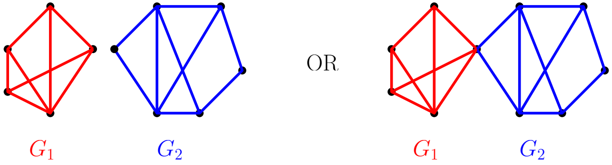

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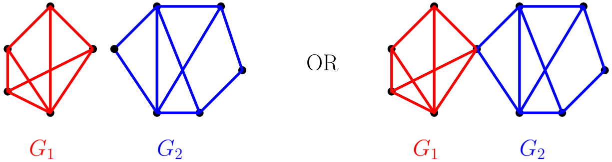

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Lemma 3.2

If multigraphs G_1 and G_2 have at most one vertex and no edges in common, then $T_{G_1 \cup G_2} = T_{G_1} T_{G_2}$.

Proof (outline). If e is neither a bridge nor a loop of G, then it is neither a bridge nor a loop of G_2, and so

$$
\begin{aligned}
& T_G(x, y) \\
= \; & T_{G-e}(x, y) + T_{G/e}(x, y) && \text{by Lemma 3.1} \\
= \; & T_{G_1 \cup (G_2 - e)}(x, y) + T_{G_1 \cup (G_2 / e)}(x, y) \\
= \; & T_{G_1}(x, y) T_{G_2 - e}(x, y) + T_{G_1}(x, y) T_{G_2 / e}(x, y) && \text{by the ind. hyp.} \\
= \; & T_{G_1}(x, y) \Big(T_{G_2 - e}(x, y) + T_{G/e}(x, y) \Big) \\
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Other cases: Lecture Notes.

Definition

A *block* of a multigraph G is a maximal connected subgraph of G that has no cut-vertices.[a]

[a] A loop with its unique endpoint is considered a block.

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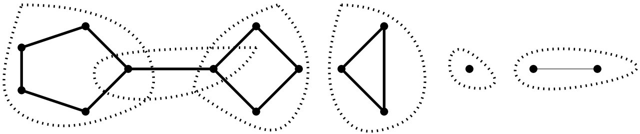
- For example, the (disconnected) graph below has six blocks, in dotted bags.



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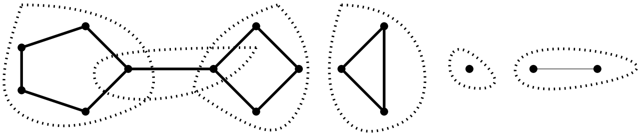

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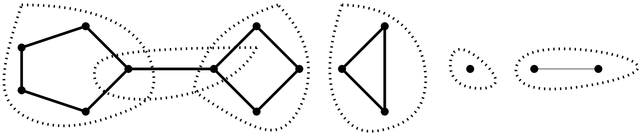

- Note that a (multi)graph can be built from its blocks by repeatedly taking disjoint unions and gluing along single vertices.
- Lemma 3.2 guarantees that the Tutte polynomial of a multigraph G is the product of the Tutte polynomials of its blocks.

Lemma 4.1

Every multigraph G satisfies

$$\pi_G(x) \;\; = \;\; (-1)^{|V(G)|-k(G)} \; x^{k(G)} \; T_G(1-x,0).$$

Proof (outline).

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- $k(G) = |V(G)|$, and so $(-1)^{|V(G)|-k(G)} = 1$.

But now it is clear that

$$\begin{aligned}\pi_G(x) &= x^{|V(G)|} \\ &= (-1)^{|V(G)|-k(G)} \cdot x^{|V(G)|} \cdot 1 \\ &= (-1)^{|V(G)|-k(G)}\ x^{k(G)}\ T_G(1-x,0)\end{aligned}$$

which is what we needed.

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Proof (outline, continued). From now on, we assume that G has at least one edge, say e.

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We consider the case when e is a bridge; the other two cases are similar (details: Lecture Notes).

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We consider the case when e is a bridge; the other two cases are similar (details: Lecture Notes).
Then either $G-e$ and G/e have exactly the same blocks, or $G-e$ can be obtained from G/e by adding an isolated vertex. Since $T_{K_1}(x,y)=1$, Lemma 3.2 now guarantees that $T_{G-e}=T_{G/e}$. We now compute (next slide):

Proof (outline, continued). Reminder: e is a bridge; $T_{G-e} = T_{G/e}$.
WTS $\pi_G(x) = (-1)^{|V(G)|-k(G)}\ x^{k(G)}\ T_G(1-x,0)$.

$$\begin{aligned}
\pi_G(x) &= \pi_{G-e}(x) - \pi_{G/e}(x) \\
&\overset{\text{ind. hyp.}}{=} (-1)^{|V(G-e)|-k(G-e)}\ x^{k(G-e)}\ T_{G-e}(1-x,0) - \\
&\quad -(-1)^{|V(G/e)|-k(G/e)}\ x^{k(G/e)}\ T_{G/e}(1-x,0) \\
&= (-1)^{|V(G)|-k(G)-1}\ x^{k(G)+1}\ T_{G-e}(1-x,0) - \\
&\quad -(-1)^{|V(G)|-k(G)-1}\ x^{k(G)}\ T_{G/e}(1-x,0) \\
&= (-1)^{|V(G)|-k(G)-1}\ x^{k(G)} \\
&\quad \Big(xT_{G-e}(1-x,0) - T_{G/e}(1-x,0)\Big) \\
&\overset{T_{G-e}=T_{G/e}}{=} (-1)^{|V(G)|-k(G)}\ x^{k(G)}\ (1-x)T_{G/e}(1-x,0) \\
&\overset{\text{Lemma 3.1}}{=} (-1)^{|V(G)|-k(G)}\ x^{k(G)}\ T_G(1-x,0)
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- So, if we know the Tutte polynomial of a (multi)graph, then we can easily compute the chromatic polynomial.
- However, computing the Tutte polynomial is hard!

Proposition 5.1

For all multigraphs G, $T_G(2,2) = 2^{|E(G)|}$.

Proposition 5.2

For all multigraphs G, $T_G(2,1)$ is the number of acyclic spanning subgraphs of G.

Proposition 5.3

If G is a connected multigraph, then $T_G(1,2)$ is the number of connected spanning subgraphs of G.

Proposition 5.4

If G is a connected multigraph, then $T_G(1,1)$ is the number of spanning trees of G.

- $r_G(A) := k_G(A) - k(G)$ and $c_G(A) := k_G(A) + |A| - |V(G)|$;
- $T_G(x, y) := \sum_{A \subseteq E(G)} (x-1)^{r_G(A)}(y-1)^{c_G(A)}$.

Proposition 5.1

For all multigraphs G, $T_G(2, 2) = 2^{|E(G)|}$.

Proof.

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Proposition 5.1

For all multigraphs G, $T_G(2,2) = 2^{|E(G)|}$.

Proof. By the definition of the Tutte polynomial, we have that

$$T_G(2,2) = \sum_{A \subseteq E(G)} (2-1)^{r_G(A)}(2-1)^{c_G(A)} = \sum_{A \subseteq E(G)} 1.$$

So, $T_G(2,2)$ is equal to the number of subsets A of $E(G)$, which is precisely $2^{|E(G)|}$.

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For all multigraphs G, $T_G(2,1)$ is the number of acyclic spanning subgraphs of G.

Proof.

- $r_G(A) := k_G(A) - k(G)$ and $c_G(A) := k_G(A) + |A| - |V(G)|$;
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Proposition 5.2

For all multigraphs G, $T_G(2, 1)$ is the number of acyclic spanning subgraphs of G.

Proof. By the definition of the Tutte polynomial, we have that

$$T_G(2,1) = \sum_{A \subseteq E(G)} (2-1)^{r_G(A)}(1-1)^{c_G(A)} = \sum_{A \subseteq E(G)} 0^{c_G(A)}$$

Now, $0^{c_G(A)} = 1$ if $c_G(A) = 0$, and $0^{c_G(A)} = 0$ otherwise. So, $T_G(2, 1)$ is equal to the number of subsets A of $E(G)$ s.t. $c_G(A) = 0$, i.e. $k_G(A) + |A| - |V(G)| = 0$, which is equivalent to $k_G(A) = |V(G)| - |A|$. But this last equality holds precisely when the multigraph $(V(G), A)$ is a forest. The result is now immediate.

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If G is a connected multigraph, then $T_G(1, 2)$ is the number of connected spanning subgraphs of G.

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If G is a connected multigraph, then $T_G(1,2)$ is the number of connected spanning subgraphs of G.

Proof. Let G be a connected multigraph. Then by the definition of the Tutte polynomial, we have that

$$T_G(1,2) = \sum_{A \subseteq E(G)} (1-1)^{r_G(A)} (2-1)^{c_G(A)} = \sum_{A \subseteq E(G)} 0^{r_G(A)}$$

Now, $0^{r_G(A)} = 1$ if $r_G(A) = 0$, and $0^{r_G(A)} = 0$ otherwise. So, $T_G(1,2)$ is equal to the number of subsets A of $E(G)$ s.t. $r_G(A) = 0$, i.e. $k_G(A) - k(G) = 0$. Since G is connected, we have that $k(G) = 1$, and so $T_G(1,2)$ is equal to the number of subsets A of $E(G)$ s.t. $k_G(A) = 1$, i.e. to the number of connected spanning subgraphs of G.

- $r_G(A) := k_G(A) - k(G)$ and $c_G(A) := k_G(A) + |A| - |V(G)|$;
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If G is a connected multigraph, then $T_G(1,1)$ is the number of spanning trees of G.

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Now, $0^{r_G(A)+c_G(A)} = 1$ if $r_G(A) + c_G(A) = 0$, and $0^{r_G(A)+c_G(A)} = 0$ otherwise. So, $T_G(1,1)$ is the number of subsets A of $E(G)$ s.t. $r_G(A) = c_G(A) = 0$. But $r_G(A) + c_G(A) = 0$ iff the multigraph $(V(G), A)$ is connected and acyclic (as in the proof of Propositions 5.2 and 5.3). So, $r_G(A) = c_G(A) = 0$ iff $(V(G), A)$ is a tree (equivalently: a spanning tree of G).

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