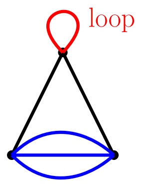
NDMI012: Combinatorics and Graph Theory 2

Lecture #9
The Tutte polynomial

Irena Penev

1 Multigraphs

A *multigraph* is an ordered pair $G = (V(G), E(G))$ such that $V(G)$ and $E(G)$ are finite sets (called the *vertex set* and *edge set*, respectively), and each edge (i.e. element of $E(G)$) is associated with two (possibly identical) vertices (i.e. elements of $V(G)$), called its *endpoints*. If an edge has only one endpoint (i.e. its two endpoints are the same), then this edge is called a *loop*. If two distinct edges have the same endpoints, then those edges are *parallel*. An edge is *incident* with a vertex, if that vertex is an endpoint of the edge. The *degree* of a vertex in a multigraph is the number of edges that it is incident with, with loops counting twice. (In the example below, all vertices are of degree four.) A multigraph is *loopless* if it has no loops.

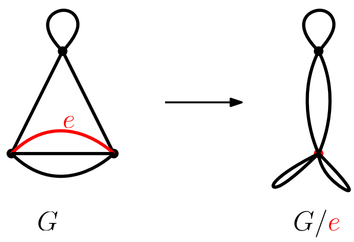


A *proper (vertex) coloring* of a loopless multigraph G is an assignment of colors to the vertices of G in such a way that, whenever two distinct vertices are joined by an edge (i.e. are the endpoints of the same edge), they receive different colors. If a multigraph has a loop, then it has no proper colorings.[1] A *k-coloring* of a multigraph G is a proper coloring of G that uses colors $1, \dots, k$ (not all of these colors need be used).

For an edge e of a multigraph G, we denote by $G - e$ the multigraph obtained by deleting e from G.

[1]Here, the idea is that if e is a loop, then its unique endpoint is adjacent to itself.

If e is a non-loop edge of a multigraph G, then the multigraph G/e obtained by *contracting* e is the multigraph obtained by first deleting e, and then identifying its endpoints to a single vertex. (Note that edges parallel to e become loops, and it is also possible that new parallel edges are created). An example is shown below.



The topic of this lecture are graph polynomials, or more precisely, multigraph polynomials (for recursive purposes, it is convenient to allow loops and parallel edges). There are a number of such polynomials. Here, we consider two: the chromatic polynomial and the Tutte polynomial.

2 The chromatic polynomial

Lemma 2.1. *For each multigraph G, there exists a unique polynomial π_G (with integer coefficients) of degree at most $|V(G)|$ such that for any non-negative integer k, $\pi_G(k)$ is the number of k-colorings of G.*

Proof. We proceed by induction on the number of edges. Fix a multigraph G, and assume inductively that the lemma is true for multigrpahs with fewer than $|E(G)|$ edges.[2]

Uniqueness follows immediately from the fact that for any non-negative integer d, any two polynomials of degree at most d that agree on at least $d+1$ points are identical.[3] It remains to prove existence. If G is edgeless, then $\pi_G(x) = x^{|V(G)|}$ is the polynomial that we need.[4] If G has at least one loop, then $\pi_G(x) = 0$ is the polynomial we need.[5]

From now on, we assume that G is loopless and has at least one edge, say e. The induction hypothesis applied to $G - e$ and G/e yields polynomials

[2]So, we assume inductively that for all multigraphs G' such that $|E(G')| < |E(G)|$, there exists a unique polynomial $\pi_{G'}$ of degree at most $|V(G')|$ such that for any non-negative integer k, $\pi_{G'}(k)$ is the number of k-colorings of G'.

[3]Indeed, suppose p_1 and p_2 are polynomials of degree at most $|V(G)|$ such that for any non-negative integer k, $p_1(k) = p_2(k)$ is the number of k-colorings of G. But then p_1 and p_2 agree on infinitely many points, and in particular, they agree on at least $|V(G)|+1$ points. So, p_1 and p_2 are identical.

[4]Indeed, if G is edgeless, then for any non-negative integer k, there are $k^{|V(G)|}$ many k-colorings of G (we simply assign colors from the set $\{1, \dots, k\}$ independently to the vertices of G).

[5]Indeed, if G has at least one loop, then G has no proper colorings.

π_{G-e} and $\pi_{G/e}$ of degree at most $|V(G)|$, and having the desired properties. Set

$$\pi_G \quad := \quad \pi_{G-e} - \pi_{G/e}.$$

Since π_{G-e} and $\pi_{G/e}$ are of degree at most $|V(G)|$, so is π_G. Now, fix a non-negative integer k. We must show that there are precisely $\pi_G(k)$ many k-colorings of G. Clearly, every k-coloring of G is also a proper coloring of $G - e$. On the other hand, a k-coloring of $G - e$ is a k-coloring of G if and only if the two endpoints of e have different colors. Further, k-colorings of $G - e$ in which both endpoints of e receive the same color correspond to k-colorings of G/e in the natural way. So, the number of k-colorings of G is equal to $\pi_{G-e}(k) - \pi_{G/e}(k) = \pi_G(k)$, which is what we needed. □

The *chromatic polynomial* of a multigraph G is the polynomial π_G from the statement of Lemma 2.1. Note that the proof of that lemma in fact gives us a recursive formula for π_G, as follows:

- if G is edgeless, then $\pi_G(x) = x^{|V(G)|}$;
- if G has a loop, then $\pi_G(x) = 0$;
- if G is loopless and has at least one edge, say e, then

$$\pi_G(x) \quad = \quad \pi_{G-e}(x) - \pi_{G/e}(x).$$

Note that $G - e$ and G/e have fewer edges than G, and so our formula really is recursive.

We remark that if G is a loopless multigraph, then $\chi(G)$ is equal to the smallest non-negative integer k such that $\pi_G(k) \neq 0$. Note that this implies that computing the chromatic polynomial is NP-hard. However, in some special cases, the chromatic polynomial is easy to compute. For example:

- $\pi_{K_n}(x) = x(x-1)(x-2)\dots(x-n+1)$;
- $\pi_T(x) = x(x-1)^{n-1}$, for any tree T on n vertices.

3 The Tutte polynomial

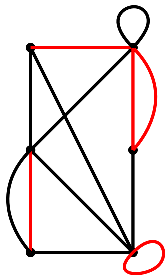
For a multigraph G, let $k(G)$ be the number of components of G; for a set $A \subseteq E(G)$, let $k_G(A)$ be the number of components of the multigraph on vertex set $V(G)$ and edge set A. Note that

$$k_G(A) \quad \geq \quad \max\{k(G), |V(G)| - |A|\},$$

and set $r_G(A) := k_G(A) - k(G)$ and $c_G(A) := k_G(A) + |A| - |V(G)|$.[6] For example, in the multigraph below (with the edges of A in red), we have

[6]Note that $r_G(A)$ and $c_G(A)$ are both non-negative.

that $k(G) = 1$, $k_G(A) = 3$, $|A| = 5$, and $|V(G)| = 6$; so, $r_G(A) = 2$ and $c_G(A) = 2$.



Now, the *Tutte polynomial* $T_G(x, y)$ of a multigraph G is defined by

$$T_G(x, y) \quad := \sum_{A \subseteq E(G)} (x-1)^{r_G(A)}(y-1)^{c_G(A)}.$$

As we shall see, the Tutte polynomial is more "general" than the chromatic polynomial, i.e. if we know the Tutte polynomial, we can easily compute the chromatic polynomial (see section 4 below). Since it is NP-hard to compute the chromatic polynomial, it is NP-hard to compute the Tutte polynomial.

Clearly, if G is edgeless, then $T_G(x, y) = 1$. Otherwise, we can get a recursive formula for $T_G(x, y)$, as follows. (A *bridge* in a multigraph G is an edge e of G such that $G - e$ has more components than G.)

Lemma 3.1. *Let e be an edge of a multigraph G. Then*

$$T_G(x, y) \quad = \begin{cases} xT_{G/e}(x, y) & \text{if } e \text{ is a bridge of } G \\ yT_{G-e}(x, y) & \text{if } e \text{ is a loop of } G \\ T_{G-e}(x, y) + T_{G/e}(x, y) & \text{otherwise} \end{cases}$$

Proof.

Claim 1. If e is a bridge of G, then $T_G(x, y) = xT_{G/e}(x, y)$.

Proof of Claim 1. Assume that e is a bridge of G. Then for any $A \subseteq E(G) \backslash \{e\}$, we have the following:

(1) $r_G(A) - 1 \overset{(*)}{=} r_G(A \cup \{e\}) \overset{(**)}{=} r_{G/e}(A)$,

(2) $c_G(A \cup \{e\}) \overset{(*)}{=} c_G(A) \overset{(**)}{=} c_{G/e}(A)$,

where, in both (1) and (2), (*) follows from the fact that e is a bridge of G, and (**) follows from the fact that contracting an edge does not change the number of components. We now compute:

$$
\begin{aligned}
& T_G(x,y) \\
= \quad & \sum_{A\subseteq E(G)} (x-1)^{r_G(A)}(y-1)^{c_G(A)} \\
= \quad & \sum_{A\subseteq E(G)\setminus\{e\}} \Big((x-1)^{r_G(A)}(y-1)^{c_G(A)} + (x-1)^{r_G(A\cup\{e\})}(y-1)^{c_G(A\cup\{e\})}\Big) \\
\overset{(1)\&(2)}{=} \quad & \sum_{A\subseteq E(G/e)} \Big((x-1)^{r_{G/e}(A)+1}(y-1)^{c_{G/e}(A)} + (x-1)^{r_{G/e}(A)}(y-1)^{c_{G/e}(A)}\Big) \\
= \quad & x\sum_{A\subseteq E(G/e)} (x-1)^{r_{G/e}(A)}(y-1)^{c_{G/e}(A)} \\
= \quad & xT_{G/e}(x,y).
\end{aligned}
$$

This proves Claim 1. ■

Claim 2. If e is a loop of G, then $T_G(x,y) = yT_{G-e}(x,y)$.

Proof of Claim 2. Assume that e is a loop of G. Deleting e does not affect the number of components, and so for each $A \subseteq E(G) \setminus \{e\}$, we have the following:

(1) $r_G(A) = r_G(A \cup \{e\}) = r_{G-e}(A)$,

(2) $c_G(A \cup \{e\}) - 1 = c_G(A) = c_{G-e}(A)$.

We now compute:

$$
\begin{aligned}
& T_G(x,y) \\
= \quad & \sum_{A\subseteq E(G)} (x-1)^{r_G(A)}(y-1)^{c_G(A)} \\
= \quad & \sum_{A\subseteq E(G)\setminus\{e\}} \Big((x-1)^{r_G(A)}(y-1)^{c_G(A)} + (x-1)^{r_G(A\cup\{e\})}(y-1)^{c_G(A\cup\{e\})}\Big) \\
\overset{(1)\&(2)}{=} \quad & \sum_{A\subseteq E(G-e)} \Big((x-1)^{r_{G-e}(A)}(y-1)^{c_{G-e}(A)} + (x-1)^{r_{G-e}(A)}(y-1)^{c_{G-e}(A)+1}\Big) \\
= \quad & y\sum_{A\subseteq E(G-e)} (x-1)^{r_{G-e}(A)}(y-1)^{c_{G-e}(A)} \\
= \quad & yT_{G-e}(x,y).
\end{aligned}
$$

This proves Claim 2. ■

Claim 3. If e is neither a bridge nor a loop of G, then $T_G(x, y) = T_{G-e}(x, y) + T_{G/e}(x, y)$.

Proof of Claim 3. Assume that e is neither a bridge nor a loop of G. Then $k(G-e) = k(G/e) = k(G)$, and it follows that for all $A \subseteq E(G)$, we have the following:

(1) $r_G(A) = r_{G-e}(A)$,

(2) $r_G(A \cup \{e\}) = r_{G/e}(A)$,

(3) $c_G(A) = c_{G-e}(A)$,

(4) $c_G(A \cup \{e\}) = c_{G/e}(A)$.

We now compute:

$$
\begin{aligned}
& T_G(x, y) \\
= \quad & \sum_{A \subseteq E(G)} (x-1)^{r_G(A)}(y-1)^{c_G(A)} \\
= \quad & \sum_{A \subseteq E(G) \setminus \{e\}} \Big((x-1)^{r_G(A)}(y-1)^{c_G(A)} + (x-1)^{r_G(A \cup \{e\})}(y-1)^{c_G(A \cup \{e\})} \Big) \\
\overset{(1)-(4)}{=} \quad & \sum_{A \subseteq E(G) \setminus \{e\}} \Big((x-1)^{r_{G-e}(A)}(y-1)^{c_{G-e}(A)} + (x-1)^{r_{G/e}(A)}(y-1)^{c_{G/e}(A)} \Big) \\
= \quad & T_{G-e}(x, y) + T_{G/e}(x, y).
\end{aligned}
$$

This proves Claim 3. ■

By Claims 1, 2, and 3, we are done. □

Further, it turns out that the Tutte polynomial is "multiplicative" in a certain sense, as the following lemma shows.

Lemma 3.2. *If multigraphs G_1 and G_2 have at most one vertex and no edges in common, then $T_{G_1 \cup G_2} = T_{G_1} T_{G_2}$.*

Proof. We prove this by induction on the number of edges, using Lemma 3.1. So, fix multigraphs G_1 and G_2 that have at most one vertex and no edges in common, set $G := G_1 \cup G_2$, and assume inductively that the lemma is true for multigraphs with fewer than $|E(G)|$ edges.[7] If G is edgeless then so are G_1 and G_2, and we have that $T_G(x, y) = T_{G_1}(x, y) = T_{G_2}(x, y) = 1$,

[7] So, we are assuming inductively that for all multigraphs G_1' and G_2' that have at most one vertex and no edges in common, if the multigraph $G_1' \cup G_2'$ has fewer than $|E(G)|$ edges, then $T_{G_1' \cup G_2'} = T_{G_1'} T_{G_2'}$.

and we are done. So, we may assume that G has at least one edge, say e. By symmetry, we may assume that $e \in E(G_2)$. Note that this means that $G - e = G_1 \cup (G_2 - e)$ and (if e is not a loop) $G/e = G_1 \cup (G_2/e)$.

Suppose first that e is a bridge of G (and therefore of G_2 as well). Then

$$\begin{aligned}
T_G(x,y) &= xT_{G/e}(x,y) && \text{by Lemma 3.1} \\
&= xT_{G_1 \cup (G_2/e)}(x,y) \\
&= xT_{G_1}(x,y)T_{G_2/e}(x,y) && \text{by the induction hypothesis} \\
&= T_{G_1}(x,y)\Big(xT_{G_2/e}(x,y)\Big) \\
&= T_{G_1}(x,y)T_{G_2}(x,y) && \text{by Lemma 3.1}
\end{aligned}$$

Next, suppose that e is a loop of G (and therefore of G_2 as well). Then

$$\begin{aligned}
T_G(x,y) &= yT_{G-e}(x,y) && \text{by Lemma 3.1} \\
&= yT_{G_1 \cup (G_2-e)}(x,y) \\
&= yT_{G_1}(x,y)T_{G_2-e}(x,y) && \text{by the induction hypothesis} \\
&= T_{G_1}(x,y)\Big(yT_{G_2-e}(x,y)\Big) \\
&= T_{G_1}(x,y)T_{G_2}(x,y) && \text{by Lemma 3.1}
\end{aligned}$$

Finally, suppose that e is neither a bridge nor a loop of G; then e is an edge of G_2 that is neither a bridge nor a loop of G_2. Then

$$\begin{aligned}
T_G(x,y) &= T_{G-e}(x,y) + T_{G/e}(x,y) && \text{by Lemma 3.1} \\
&= T_{G_1 \cup (G_2-e)}(x,y) + T_{G_1 \cup (G_2/e)}(x,y) \\
&= T_{G_1}(x,y)T_{G_2-e}(x,y) + T_{G_1}(x,y)T_{G_2/e}(x,y) && \text{by the ind. hyp.} \\
&= T_{G_1}(x,y)\Big(T_{G_2-e}(x,y) + T_{G/e}(x,y)\Big) \\
&= T_{G_1}(x,y)T_{G_2}(x,y) && \text{by Lemma 3.1}
\end{aligned}$$

This completes the argument. □

Note that Lemma 3.2 guarantees that the Tutte polynomial of a multigraph G is the product of the Tutte polynomials of its blocks.[8]

[8] A *block* of a multigraph G is a maximal connected subgraph of G that has no cut-

4 The relationship between the chromatic polynomial and the Tutte polynomial

As our next lemma shows, the Tutte polynomial is more general than the chromatic polynomial, i.e. if we know the Tutte polynomial of a multigraph, we can easily compute the chromatic polynomial of that multigraph.

Lemma 4.1. *Every multigraph G satisfies*

$$\pi_G(x) \quad = \quad (-1)^{|V(G)|-k(G)}\ x^{k(G)}\ T_G(1-x, 0).$$

Proof. We proceed by induction on the number of edges. Fix a multigraph G, and assume inductively that the statement is true for all multigraphs on fewer than $|E(G)|$ edges.

Suppose first that G is edgeless. Then by section 2, we have that $\pi_G(x) = x^{|V(G)|}$. Further, by the definition of the Tutte polynomial, we have that $T_G(x, y) = 1$, and so $T_G(1-x, 0) = 1$. Moreover, $k(G) = |V(G)|$, and so $(-1)^{|V(G)|-k(G)} = 1$. But now it is clear that

$$\begin{aligned}
\pi_G(x) &= x^{|V(G)|} \\
&= (-1)^{|V(G)|-k(G)} \cdot x^{|V(G)|} \cdot 1 && \text{because } (-1)^{|V(G)|-k(G)} = 1 \\
&= (-1)^{|V(G)|-k(G)}\ x^{k(G)}\ T_G(1-x, 0) && \text{because } T_G(1-x, 0) = 1
\end{aligned}$$

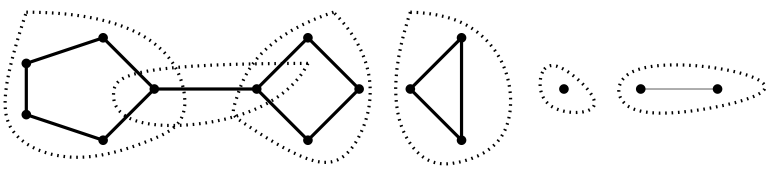
which is what we needed.

From now on, we assume that G has at least one edge, say e. We consider three cases: when e is a bridge, when e is a loop, and when e is neither a bridge nor a loop.

Suppose first that e is a bridge of G. Then either $G - e$ and G/e have exactly the same blocks, or $G - e$ can be obtained from G/e by adding

---

vertices. (However, not all such subgraphs are blocks! We need maximality.) For example, the (disconnected) graph below has six blocks, in dotted bags.



Note that a (multi)graph can be built from its blocks by repeatedly taking disjoint unions and gluing along single vertices. In the case of graphs (with no loops and no parallel edges), blocks are the maximal 2-connected subgraphs, bridges (with their endpoints), and components on at most two vertices. In the multigraph case, a loop (with its unique endpoint) is considered a block.

an isolated vertex. Since $T_{K_1}(x,y) = 1$, Lemma 3.2 now guarantees that $T_{G-e} = T_{G/e}$. We now compute:

$$\begin{aligned}
\pi_G(x) &= \pi_{G-e}(x) - \pi_{G/e}(x) && \text{by section 2} \\
&= (-1)^{|V(G-e)|-k(G-e)}\ x^{k(G-e)}\ T_{G-e}(1-x,0) - \\
&\quad -(-1)^{|V(G/e)|-k(G/e)}\ x^{k(G/e)}\ T_{G/e}(1-x,0) && \text{by the ind. hyp.} \\
&= (-1)^{|V(G)|-k(G)-1}\ x^{k(G)+1}\ T_{G-e}(1-x,0) - \\
&\quad -(-1)^{|V(G)|-k(G)-1}\ x^{k(G)}\ T_{G/e}(1-x,0) \\
&= (-1)^{|V(G)|-k(G)-1}\ x^{k(G)} \\
&\quad \Big(xT_{G-e}(1-x,0) - T_{G/e}(1-x,0) \Big) \\
&= (-1)^{|V(G)|-k(G)}\ x^{k(G)}\ (1-x)T_{G/e}(1-x,0) && \text{because } T_{G-e} = T_{G/e} \\
&= (-1)^{|V(G)|-k(G)}\ x^{k(G)}\ T_G(1-x,0) && \text{by Lemma 3.1,}
\end{aligned}$$

which is what we needed.

Next, suppose that e is a loop. Then by section 2, $\pi_G(x) = 0$. On the other hand, by Lemma 3.1, we have that $T_G(x,y) = yT_{G-e}(x,y)$, and consequently, $T_G(1-x,0) = 0$. It then immediately follows that

$$\pi_G(x) = (-1)^{|V(G)|-k(G)}\ x^{k(G)}\ T_G(1-x,0).$$

Finally, suppose that e is neither a bridge nor a loop. We then compute:

$$\begin{aligned}
\pi_G(x) &= \pi_{G-e}(x) - \pi_{G/e}(x) && \text{by section 2} \\
&= (-1)^{|V(G-e)|-k(G-e)}\ x^{k(G-e)}\ T_{G-e}(1-x,0) - \\
&\quad -(-1)^{|V(G/e)|-k(G/e)}\ x^{k(G/e)}\ T_{G/e}(1-x,0) && \text{by the ind. hyp.} \\
&= (-1)^{|V(G)|-k(G)}\ x^{k(G)}\ T_{G-e}(1-x,0) - \\
&\quad -(-1)^{|V(G)|-k(G)-1}\ x^{k(G)}\ T_{G/e}(1-x,0) \\
&= (-1)^{|V(G)|-k(G)}\ x^{k(G)}\ T_{G-e}(1-x,0) + \\
&\quad +(-1)^{|V(G)|-k(G)}\ x^{k(G)}\ T_{G/e}(1-x,0) \\
&= (-1)^{|V(G)|-k(G)}\ x^{k(G)} \\
&\quad \Big(T_{G-e}(1-x,0) + T_{G/e}(1-x,0) \Big) \\
&= (-1)^{|V(G)|-k(G)}\ x^{k(G)}\ T_G(1-x,0) && \text{by Lemma 3.1}
\end{aligned}$$

which is what we needed. This completes the argument. □

5 Some special points of the Tutte polynomial

In this section, we give a combinatorial interpretation of the Tutte polynomial evaluated at some special points.

Proposition 5.1. *For all multigraphs G, $T_G(2,2) = 2^{|E(G)|}$.*

Proof. By the definition of the Tutte polynomial, we have that

$$T_G(2,2) = \sum_{A \subseteq E(G)} (2-1)^{r_G(A)}(2-1)^{c_G(A)} = \sum_{A \subseteq E(G)} 1.$$

So, $T_G(2,2)$ is equal to the number of subsets A of $E(G)$, which is precisely $2^{|E(G)|}$. □

A *spanning subgraph* of a multigraph G is a multigraph H such that $V(H) = V(G)$ and $E(H) \subseteq E(G)$. A multigraph is *acyclic* if it has no cycles; in particular, acyclic multigraphs have no loops and no parallel edges, and so an acyclic (multi)graph is simply a forest.

Proposition 5.2. *For all multigraphs G, $T_G(2,1)$ is the number of acyclic spanning subgraphs of G.*[9]

Proof. By the definition of the Tutte polynomial, we have that

$$T_G(2,1) = \sum_{A \subseteq E(G)} (2-1)^{r_G(A)}(1-1)^{c_G(A)} = \sum_{A \subseteq E(G)} 0^{c_G(A)}$$

Now, $0^{c_G(A)} = 1$ if $c_G(A) = 0$, and $0^{c_G(A)} = 0$ otherwise. So, $T_G(2,1)$ is equal to the number of subsets A of $E(G)$ such that $c_G(A) = 0$, i.e. $k_G(A) + |A| - |V(G)| = 0$, which is equivalent to $k_G(A) = |V(G)| - |A|$. But this last equality holds precisely when the multigraph $(V(G), A)$ is a forest. The result is now immediate. □

Proposition 5.3. *If G is a connected multigraph, then $T_G(1,2)$ is the number of connected spanning subgraphs of G.*

Proof. Let G be a connected multigraph. Then by the definition of the Tutte polynomial, we have that

$$T_G(1,2) = \sum_{A \subseteq E(G)} (1-1)^{r_G(A)}(2-1)^{c_G(A)} = \sum_{A \subseteq E(G)} 0^{r_G(A)}$$

Now, $0^{r_G(A)} = 1$ if $r_G(A) = 0$, and $0^{r_G(A)} = 0$ otherwise. So, $T_G(1,2)$ is equal to the number of subsets A of $E(G)$ such that $r_G(A) = 0$, i.e. $k_G(A) - k(G) = 0$. Since G is connected, we have that $k(G) = 1$, and so $T_G(1,2)$ is equal to the number of subsets A of $E(G)$ such that $k_G(A) = 1$, i.e. to the number of connected spanning subgraphs of G. □

[9] As a terminological matter, a spanning acyclic subgraph is not quite the same thing as a spanning forest. The term "spanning forest" is generally reserved for forests whose components are spanning trees of the components of the original (multi)graph, which is a more restricted notion. So, $T_G(2,1)$ need **not** be the number of spanning forests of G.

Proposition 5.4. *If G is a connected multigraph, then $T_G(1,1)$ is the number of spanning trees of G.*

Proof. Let G be a connected multigraph. Then by the definition of the Tutte polynomial, we have that

$$T_G(1,1) \quad = \sum_{A \subseteq E(G)} (1-1)^{r_G(A)}(1-1)^{c_G(A)} \quad = \sum_{A \subseteq E(G)} 0^{r_G(A)+c_G(A)}$$

Now, $0^{r_G(A)+c_G(A)} = 1$ if $r_G(A)+c_G(A) = 0$, and $0^{r_G(A)+c_G(A)} = 0$ otherwise. So, $T_G(1,1)$ is the number of subsets A of $E(G)$ such that $r_G(A) = c_G(A) = 0$. But $r_G(A) = 0$ if and only if the multigraph $(V(G), A)$ is connected (as in the proof of Proposition 5.3), and $c_G(A) = 0$ if the multigraph $(V(G), A)$ is if and only if acyclic (as in the proof of Proposition 5.2). So, $r_G(A) = c_G(A) = 0$ if and only if $(V(G), A)$ is a tree (equivalently: a spanning tree of G). □