NDMI012: Combinatorics and Graph Theory 2

Lecture #8

Perfect graphs

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April 5, 2022

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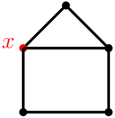
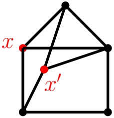
- The Perfect Graph Theorem was originally conjectured by Berge (1961).
- Before it was proven, the Perfect Graph Theorem was known as the Weak Perfect Graph Conjecture.
- We will prove the theorem, but first we need some terminology and a lemma.

- *Duplicating a vertex* x of a graph G produces a supergraph $G \circ x$ by adding to G a vertex x' and making it adjacent to all the neighbors of x in G, and to no other vertices of G (in particular, x and x' are nonadjacent in $G \circ x$).

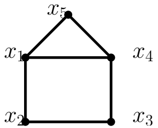


G



$G \circ x$

- *Vertex multiplication* of a graph G with vertex set $V(G) = \{x_1, \ldots, x_n\}$ by a nonnegative integer vector $h = (h_1, \ldots, h_n)$ is the graph $G \circ h$ having h_i pairwise nonadjacent copies of x_i, such that copies of x_i and x_j are adjacent in $G \circ x$ if and only if $x_i x_j \in E(G)$.



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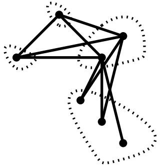
$h = (1, 0, 3, 2, 1)$



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  - A graph G is *χ-perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$.
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  - A graph G is *α-perfect* if every induced subgraph H of G satisfies $\overline{\chi}(H) = \alpha(H)$.
- Obviously, a graph is χ-perfect (i.e. perfect) if and only if its complement is α-perfect.

Lemma 1.1 [Berge, 1961]

Vertex multiplication preserves χ-perfection and α-perfection.[a]

[a] This means that for every graph G with vertex set $V(G) = \{x_1, \ldots, x_n\}$, and every nonnegative integer vector $h = (h_1, \ldots, h_n)$, we have the following:

- if G is χ-perfect, then so is $G \circ h$;
- if G is α-perfect, then so is $G \circ h$.

Proof.

Claim 1. *Vertex duplication preserves χ-perfection.*

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Clearly, we can extend an optimal coloring of G to a proper coloring of $G \circ x$, by giving x' the same color as x. So, $\chi(G \circ x) = \chi(G)$.

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Since G is χ-perfect, we have that $\chi(G) = \omega(G)$, and we now see that $\chi(G \circ x) = \chi(G) = \omega(G) = \omega(G \circ x)$. This proves Claim 1.

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Suppose first that x belongs to a maximum stable set of G. Then $\alpha(G \circ x) = \alpha(G) + 1$. Since $\overline{\chi}(G) = \alpha(G)$ (because G is α-perfect), we can obtain a clique cover of size $\alpha(G) + 1$ by adding $\{x'\}$ as a one-vertex clique to some set of $\overline{\chi}(G)$ many cliques covering G. This is enough because now we have that $\overline{\chi}(G) + 1 = \alpha(G) + 1 = \alpha(G \circ x) \leq \overline{\chi}(G \circ x) \leq \overline{\chi}(G) + 1$, and so $\overline{\chi}(G \circ x) = \alpha(G \circ x)$.

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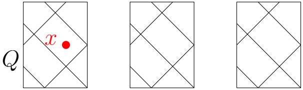
Proof of Claim 2 (continued). We may now assume that x does not belong to any maximum stable set of G. Then $\alpha(G \circ x) = \alpha(G)$. Let Q be the clique containing x in a minimum clique cover of G.



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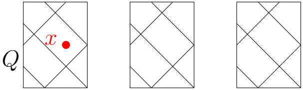

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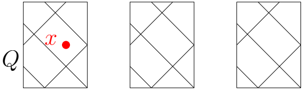

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$\alpha(G \setminus Q') = \alpha(G) - 1$.

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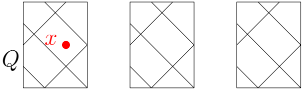

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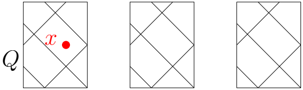

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$\alpha(G) = \alpha(G \circ x)$ many cliques covering $G \circ x$; we now have that
$\overline{\chi}(G \circ x) = \alpha(G \circ x)$. This proves Claim 2.

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Proof (continued). So far, we have proven:

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Now, $G \circ h$ can be obtained from $G[A]$ by a sequence of vertex duplications: if every h_i is 0 or 1 then $G \circ h = G[A]$, and otherwise, $G \circ h$ can be obtained from $G[A]$ by repeatedly duplicating vertices until there are h_i copies of each x_i.

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Since vertex duplication preserves χ-perfection and α-perfection (by Claims 1 and 2), an easy induction now guarantees that if G is χ-perfect (resp. α-perfect), then $G \circ h$ is also χ-perfect (resp. α-perfect). This completes the argument.

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$$G \text{ is } (\chi\text{-})\text{perfect} \implies \overline{G} \text{ is } \alpha\text{-perfect} \implies \overline{G} \text{ is } (\chi\text{-})\text{perfect},$$

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Now, fix an α-perfect graph G, and assume inductively that all α-perfect graphs on fewer than $|V(G)|$ vertices are χ-perfect. We must show that G is χ-perfect.

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Suppose first that G has a stable set S that intersects every maximum clique of G. Then by the minimality of G, $\chi(G \setminus S) = \omega(G \setminus S) = \omega(G) - 1$.

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From now on, we assume that every stable set S of G misses (i.e. has an empty intersection with) some maximum clique $Q(S)$; our goal is to derive a contradiction. Set $V(G) = \{x_1, \dots, x_n\}$, and let $\mathcal{S} = \{S_1, \dots, S_t\}$ be the set of all maximal stable sets of G.

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From now on, we assume that every stable set S of G misses (i.e. has an empty intersection with) some maximum clique $Q(S)$; our goal is to derive a contradiction. Set $V(G) = \{x_1, \dots, x_n\}$, and let $\mathcal{S} = \{S_1, \dots, S_t\}$ be the set of all maximal stable sets of G. For every vertex x_j, let h_j be the number of stable sets S in $\mathcal{S}$ such that $x_j \in Q(S)$.

The Perfect Graph Theorem [Lovász, 1972]

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Let $A = [a_{i,j}]_{t \times n}$ be a 0,1-matrix of the incidence relation between the set of $Q(S)$'s for $S \in \mathcal{S}$ and $V(G)$. So, $a_{i,j} = 1$ if and only if $x_j \in Q(S_i)$.

$$\begin{array}{cc} & \begin{matrix} x_1 & \dots x_j & \dots x_n \end{matrix} \\ \begin{matrix} Q(S_1) \\ \vdots \\ Q(S_i) \\ \vdots \\ Q(S_t) \end{matrix} & \begin{bmatrix} & & \\ & & \\ & a_{i,j} & \\ & & \\ & & \end{bmatrix} \end{array}$$

By construction, h_j is the number of 1's in column j of A, and $|V(H)|$ is the total number of 1's in A.

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Since each row contributes $\omega(G)$ ones, we have $|V(H)| = \omega(G)|\mathcal{S}|$.

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Since each row contributes $\omega(G)$ ones, we have $|V(H)| = \omega(G)|\mathcal{S}|$. Since vertex duplication cannot enlarge cliques, we have $\omega(H) \leq \omega(G)$.

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Since each row contributes $\omega(G)$ ones, we have $|V(H)| = \omega(G)|\mathcal{S}|$. Since vertex duplication cannot enlarge cliques, we have $\omega(H) \leq \omega(G)$. Therefore $\overline{\chi}(H) \geq \frac{|V(H)|}{\omega(H)} \geq \frac{|V(H)|}{\omega(G)} = |\mathcal{S}|$.

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Every stable set of H consists of copies of elements in some stable set of G; so, a maximum stable set of H consists of all copies of all vertices in some maximal stable set of G.

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$$\alpha(H) = \max_{T \in \mathcal{S}} \sum_{j : x_j \in T} h_j.$$

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The sum above counts the 1's in A that appear in the columns indexed by the vertices of T.

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$\alpha(H) = \max\limits_{T \in \mathcal{S}} \sum\limits_{S \in \mathcal{S}} |T \cap Q(S)|$.

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Since T is a stable set, it has at most one vertex in each chosen clique $Q(S)$. Also, $T \cap Q(T) = \emptyset$.

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In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

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> ***Claim 1.*** *Any antichain of size k in $(X_0, \preceq)$ intersects each of $C_1, \dots, C_k$ in exactly one element.*

Proof of Claim 1.

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Proof of Claim 1. Let B be a antichain of size k in $(X_0, \preceq)$. Since B is a antichain and $C_1, \dots, C_k$ are chains in $(X_0, \preceq)$, we see that B intersects each of $C_1, \dots, C_k$ in at most one element.

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Proof of Claim 1. Let B be a antichain of size k in $(X_0, \preceq)$. Since B is a antichain and $C_1, \dots, C_k$ are chains in $(X_0, \preceq)$, we see that B intersects each of $C_1, \dots, C_k$ in at most one element. But since $B \subseteq C_1 \cup \dots \cup C_k$, and since $|B| = k$, we see that B intersects each of $C_1, \dots, C_k$ in exactly one element. This proves Claim 1.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued).

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Proof (continued). Now, for all $i \in \{1, \ldots, k\}$, let C_i' be the set of all elements of C_i that belong to some antichain of $(X_0, \preceq)$ of size k;

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Claim 2. *$\{x_1, \dots, x_k\}$ is an antichain in $(X_0, \preceq)$.*

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Claim 2. $\{x_1, \dots, x_k\}$ *is an antichain in* $(X_0, \preceq)$.

Proof of Claim 2.

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Proof (continued). Suppose first that $\{x_0, x_1, \ldots, x_k\}$ is an antichain in $(X, \preceq)$. Then this antichain is of size $k+1$, and $\{C_1, \ldots, C_k, \{x_0\}\}$ is a chain decomposition of $(X, \preceq)$ of size $k+1$, and we are done.

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Definition

A *comparability graph* (or a *transitively orientable graph*) is a graph G such that there exists a partial order $\preceq$ on $V(G)$ such that for all distinct $x, y \in V(G)$, we have that $xy \in E(G)$ if and only if x and y are comparable with respect to $\preceq$.

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- Equivalently, G is a comparability graph if there exists an orientation $\vec{G} = (V(G), A(G))$ of G such that for all $\vec{uv}, \vec{vw} \in A(G)$, we have that $\vec{uw} \in A(G)$.

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- Note that in a comparability graph, cliques correspond to chains, and stable sets correspond to antichains.

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Corollary 2.1

Every comparability graph is perfect. The complement of any comparability graph is perfect.

Proof (outline). In view of the Perfect Graph Theorem, it suffices to show that the complement of any comparability graph is perfect. But this follows from Dilworth's theorem by an easy induction (details: Lecture Notes).

Lemma 3.1

Every bipartite graph is perfect.

Proof. Obvious.

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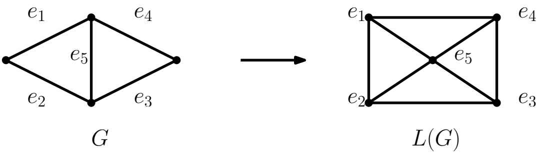
Lemma 3.2

The complement of any bipartite graph is perfect.

Proof. This follows immediately from Lemma 3.1 and the Perfect Graph Theorem.

Definition

Given a graph G, the *line graph* of G, denoted by $L(G)$, is the graph with vertex set $E(G)$, in which distinct $e, f \in E(G)$ are adjacent if and only if they share an endpoint in G.



Lemma 3.3

The line graph of any bipartite graph is perfect.

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Lemma 3.4

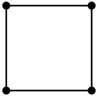
The complement of the line graph of any bipartite graph is perfect.

Proof. This follows immediately from Lemma 3.3 and the Perfect Graph Theorem.

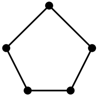
Definition

A *hole* in a graph G is an induced cycle of length at least four. An *antihole* in G is an induced subgraph H of G such that $\overline{H}$ is a hole in $\overline{G}$.

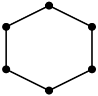
- Holes:



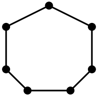
C_4



C_5

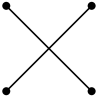


C_6

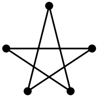


C_7

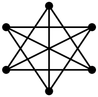
- Antiholes:



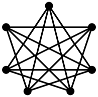
$\overline{C_4} \cong 2K_2$



$\overline{C_5} \cong C_5$



$\overline{C_6}$



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- Clearly, a graph is Berge if and only if its complement is Berge.
- So, the Strong Perfect Graph Theorem immediately implies the Perfect Graph Theorem.

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A graph is perfect if and only if it is Berge.

- One direction of the Strong Perfect Graph Theorem ("every perfect graph is Berge") is an easy exercise.

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- Indeed, it is easy to check that for each integer $n \geq 2$, we have that
  - $\omega(C_{2n+1}) = 2$ and $\chi(C_{2n+1}) = 3$;
  - $\omega(\overline{C_{2n+1}}) = n$ and $\chi(\overline{C_{2n+1}}) = n + 1$.

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    - $\omega(\overline{C_{2n+1}}) = n$ and $\chi(\overline{C_{2n+1}}) = n+1$.
- So, odd holes and antiholes are imperfect, and therefore, no perfect graph contains an odd hole or an odd antihole. Thus, every perfect graph is Berge.

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  - The proof of this decomposition theorem is by far the most complicated part of the proof of the Strong Perfect Graph Theorem, and it is over 100 pages long.
- The "basic" graphs are bipartite graphs and their complements, line graphs of bipartite graphs, complements of line graphs of bipartite graphs, and "double split" graphs (we omit the definition).

The Strong Perfect Graph Theorem [Chudnovsky, Robertson, Seymour, Thomas, 2002]

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- What about the other direction ("every Berge graph is perfect")?
- It relies on a "decomposition theorem" for Berge graphs, which, roughly, states that every Berge graph either is "basic" or admits a "decomposition."
    - The proof of this decomposition theorem is by far the most complicated part of the proof of the Strong Perfect Graph Theorem, and it is over 100 pages long.
- The "basic" graphs are bipartite graphs and their complements, line graphs of bipartite graphs, complements of line graphs of bipartite graphs, and "double split" graphs (we omit the definition).
- All basic graphs are perfect: we proved this for the first four types of basic graphs, and the proof for double split graphs is easy.

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- There are several "decompositions" (we omit the definitions), and it can be shown that no imperfect Berge graph of minimum possible size admits any of these decompositions.
- It now follows that all Berge graphs are perfect.

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  - In fact, weighted versions of these problems can also be solved in polynomial time.