NDMI012: Combinatorics and Graph Theory 2

Lecture #7

Chordal graphs

Irena Penev

March 29, 2022

This lecture consists of three parts:

This lecture consists of three parts:

1. triangle-free graphs of arbitrarily large chromatic number;

This lecture consists of three parts:

1. triangle-free graphs of arbitrarily large chromatic number;
2. a very brief introduction to perfect graphs;

This lecture consists of three parts:

1. triangle-free graphs of arbitrarily large chromatic number;
2. a very brief introduction to perfect graphs;
3. chordal graphs.

Part I: Triangle-free graphs of arbitrarily large chromatic number

Part I: Triangle-free graphs of arbitrarily large chromatic number

- Every graph G satisfies $\omega(G) \leq \chi(G)$.

Part I: Triangle-free graphs of arbitrarily large chromatic number

- Every graph G satisfies $\omega(G) \leq \chi(G)$.
- So, the simplest way to construct a graph of high chromatic number is to construct a graph that has a large clique number.

Part I: Triangle-free graphs of arbitrarily large chromatic number

- Every graph G satisfies $\omega(G) \leq \chi(G)$.
- So, the simplest way to construct a graph of high chromatic number is to construct a graph that has a large clique number.
- However, as we shall see, it is possible to construct graphs of small clique number and large chromatic number.

Part I: Triangle-free graphs of arbitrarily large chromatic number

- Every graph G satisfies $\omega(G) \leq \chi(G)$.
- So, the simplest way to construct a graph of high chromatic number is to construct a graph that has a large clique number.
- However, as we shall see, it is possible to construct graphs of small clique number and large chromatic number.

Definition

A *triangle* in a graph G is a clique of size three. A graph is *triangle-free* if it contains no triangles.

- So, a graph is triangle-free iff its clique number is ≤ 2.

Part I: Triangle-free graphs of arbitrarily large chromatic number

- Every graph G satisfies $\omega(G) \leq \chi(G)$.
- So, the simplest way to construct a graph of high chromatic number is to construct a graph that has a large clique number.
- However, as we shall see, it is possible to construct graphs of small clique number and large chromatic number.

Definition

A *triangle* in a graph G is a clique of size three. A graph is *triangle-free* if it contains no triangles.

- So, a graph is triangle-free iff its clique number is ≤ 2.
- Our goal is to construct a family of triangle-free graphs of arbitrarily large chromatic number.

Part I: Triangle-free graphs of arbitrarily large chromatic number

- Every graph G satisfies $\omega(G) \leq \chi(G)$.
- So, the simplest way to construct a graph of high chromatic number is to construct a graph that has a large clique number.
- However, as we shall see, it is possible to construct graphs of small clique number and large chromatic number.

Definition

A *triangle* in a graph G is a clique of size three. A graph is *triangle-free* if it contains no triangles.

- So, a graph is triangle-free iff its clique number is ≤ 2.
- Our goal is to construct a family of triangle-free graphs of arbitrarily large chromatic number.
- There are several known constructions; here, we give the one due to Mycielski (1955).

- The Mycielski graphs $\{M_k\}_{k=2}^{\infty}$ are defined recursively, as follows.

- The Mycielski graphs $\{M_k\}_{k=2}^{\infty}$ are defined recursively, as follows.
- First, let $M_2 = K_2$.

- The Mycielski graphs $\{M_k\}_{k=2}^{\infty}$ are defined recursively, as follows.
- First, let $M_2 = K_2$.
- Next, fix an integer k, and suppose M_k has been constructed. We construct M_{k+1} as follows.

- The Mycielski graphs $\{M_k\}_{k=2}^{\infty}$ are defined recursively, as follows.
- First, let $M_2 = K_2$.
- Next, fix an integer k, and suppose M_k has been constructed. We construct M_{k+1} as follows.
- Let $V = \{v_1, \ldots, v_n\}$ be the vertex set of M_k. Let $U = \{u_1, \ldots, u_n\}$ (where the u_i's are "new" vertices; we think of u_i as a "duplicate" of v_i), and let w be another "new" vertex.

- The Mycielski graphs $\{M_k\}_{k=2}^{\infty}$ are defined recursively, as follows.
- First, let $M_2 = K_2$.
- Next, fix an integer k, and suppose M_k has been constructed. We construct M_{k+1} as follows.
- Let $V = \{v_1, \ldots, v_n\}$ be the vertex set of M_k. Let $U = \{u_1, \ldots, u_n\}$ (where the u_i's are "new" vertices; we think of u_i as a "duplicate" of v_i), and let w be another "new" vertex.
- Let M_{k+1} have vertex set $V \cup U \cup \{w\}$ and adjacency as follows:

- The Mycielski graphs $\{M_k\}_{k=2}^{\infty}$ are defined recursively, as follows.
- First, let $M_2 = K_2$.
- Next, fix an integer k, and suppose M_k has been constructed. We construct M_{k+1} as follows.
- Let $V = \{v_1, \ldots, v_n\}$ be the vertex set of M_k. Let $U = \{u_1, \ldots, u_n\}$ (where the u_i's are "new" vertices; we think of u_i as a "duplicate" of v_i), and let w be another "new" vertex.
- Let M_{k+1} have vertex set $V \cup U \cup \{w\}$ and adjacency as follows:
    - adjacency between the v_i's is inherited from M_k, that is, $M_{k+1}[V] = M_k$;

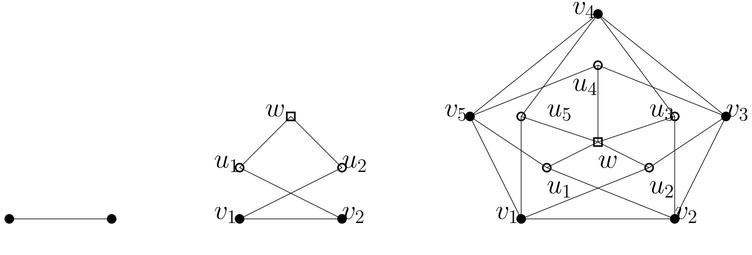
- The Mycielski graphs $\{M_k\}_{k=2}^{\infty}$ are defined recursively, as follows.
- First, let $M_2 = K_2$.
- Next, fix an integer k, and suppose M_k has been constructed. We construct M_{k+1} as follows.
- Let $V = \{v_1, \dots, v_n\}$ be the vertex set of M_k. Let $U = \{u_1, \dots, u_n\}$ (where the u_i's are "new" vertices; we think of u_i as a "duplicate" of v_i), and let w be another "new" vertex.
- Let M_{k+1} have vertex set $V \cup U \cup \{w\}$ and adjacency as follows:
    - adjacency between the v_i's is inherited from M_k, that is, $M_{k+1}[V] = M_k$;
    - for all $i \in \{1, \dots, n\}$, u_i is non-adjacent to v_i;

- The Mycielski graphs $\{M_k\}_{k=2}^{\infty}$ are defined recursively, as follows.
- First, let $M_2 = K_2$.
- Next, fix an integer k, and suppose M_k has been constructed. We construct M_{k+1} as follows.
- Let $V = \{v_1, \dots, v_n\}$ be the vertex set of M_k. Let $U = \{u_1, \dots, u_n\}$ (where the u_i's are "new" vertices; we think of u_i as a "duplicate" of v_i), and let w be another "new" vertex.
- Let M_{k+1} have vertex set $V \cup U \cup \{w\}$ and adjacency as follows:
    - adjacency between the v_i's is inherited from M_k, that is, $M_{k+1}[V] = M_k$;
    - for all $i \in \{1, \dots, n\}$, u_i is non-adjacent to v_i;
    - for all distinct $i, j \in \{1, \dots, n\}$, u_i is adjacent to v_j in M_{k+1} iff v_i is adjacent to v_j in M_k;

- The Mycielski graphs $\{M_k\}_{k=2}^{\infty}$ are defined recursively, as follows.
- First, let $M_2 = K_2$.
- Next, fix an integer k, and suppose M_k has been constructed. We construct M_{k+1} as follows.
- Let $V = \{v_1, \dots, v_n\}$ be the vertex set of M_k. Let $U = \{u_1, \dots, u_n\}$ (where the u_i's are "new" vertices; we think of u_i as a "duplicate" of v_i), and let w be another "new" vertex.
- Let M_{k+1} have vertex set $V \cup U \cup \{w\}$ and adjacency as follows:
    - adjacency between the v_i's is inherited from M_k, that is, $M_{k+1}[V] = M_k$;
    - for all $i \in \{1, \dots, n\}$, u_i is non-adjacent to v_i;
    - for all distinct $i, j \in \{1, \dots, n\}$, u_i is adjacent to v_j in M_{k+1} iff v_i is adjacent to v_j in M_k;
    - U is a stable set in M_{k+1};

- The Mycielski graphs $\{M_k\}_{k=2}^{\infty}$ are defined recursively, as follows.
- First, let $M_2 = K_2$.
- Next, fix an integer k, and suppose M_k has been constructed. We construct M_{k+1} as follows.
- Let $V = \{v_1, \dots, v_n\}$ be the vertex set of M_k. Let $U = \{u_1, \dots, u_n\}$ (where the u_i's are "new" vertices; we think of u_i as a "duplicate" of v_i), and let w be another "new" vertex.
- Let M_{k+1} have vertex set $V \cup U \cup \{w\}$ and adjacency as follows:
  - adjacency between the v_i's is inherited from M_k, that is, $M_{k+1}[V] = M_k$;
  - for all $i \in \{1, \dots, n\}$, u_i is non-adjacent to v_i;
  - for all distinct $i, j \in \{1, \dots, n\}$, u_i is adjacent to v_j in M_{k+1} iff v_i is adjacent to v_j in M_k;
  - U is a stable set in M_{k+1};
  - w is adjacent to all vertices in U and non-adjacent to all vertices in V.

- The first three Mycielski graphs are represented in the picture below.



M_2 M_3 M_4

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof. We proceed by induction on k.

Lemma 1.1

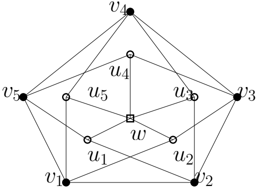
For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof. We proceed by induction on k. The lemma is clearly true for $k = 2$.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

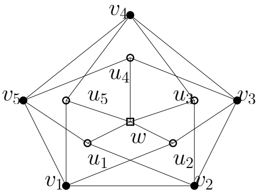
Proof. We proceed by induction on k. The lemma is clearly true for $k = 2$. Next, fix an integer $k \geq 2$, and assume inductively that $\omega(M_k) = 2$ and $\chi(M_k) = k$. WTS $\omega(M_{k+1}) = 2$ and $\chi(M_{k+1}) = k + 1$.



Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

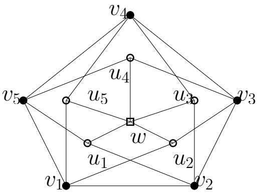
Proof (continued). WTS $\omega(M_{k+1}) = 2$.



Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof (continued). WTS $\omega(M_{k+1}) = 2$.

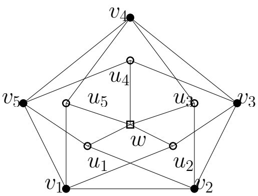


Since $\omega(M_k) = 2$, and M_k is a subgraph of M_{k+1}, it is clear that $\omega(M_{k+1}) \geq 2$.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof (continued). WTS $\omega(M_{k+1}) = 2$.



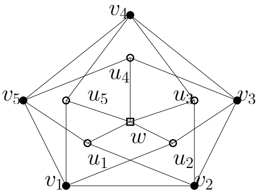
Since $\omega(M_k) = 2$, and M_k is a subgraph of M_{k+1}, it is clear that $\omega(M_{k+1}) \geq 2$.

WTS M_{k+1} is triangle-free.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof (continued). WTS $\omega(M_{k+1}) = 2$.



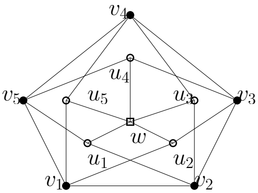
Since $\omega(M_k) = 2$, and M_k is a subgraph of M_{k+1}, it is clear that $\omega(M_{k+1}) \geq 2$.

WTS M_{k+1} is triangle-free. Suppose otherwise, and let T be a triangle in M_{k+1}.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof (continued). WTS $\omega(M_{k+1}) = 2$.



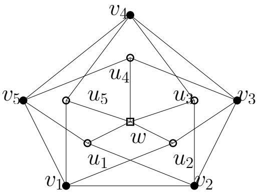
Since $\omega(M_k) = 2$, and M_k is a subgraph of M_{k+1}, it is clear that $\omega(M_{k+1}) \geq 2$.

WTS M_{k+1} is triangle-free. Suppose otherwise, and let T be a triangle in M_{k+1}. Since U is a stable set of G, we see that $|T \cap U| \leq 1$.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof (continued). WTS $\omega(M_{k+1}) = 2$.



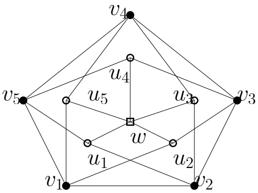
Since $\omega(M_k) = 2$, and M_k is a subgraph of M_{k+1}, it is clear that $\omega(M_{k+1}) \geq 2$.

WTS M_{k+1} is triangle-free. Suppose otherwise, and let T be a triangle in M_{k+1}. Since U is a stable set of G, we see that $|T \cap U| \leq 1$. Since $N_{M_{k+1}}(w) = U$, and since U is a stable set, we further see that $w \notin T$.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof (continued). WTS $\omega(M_{k+1}) = 2$.



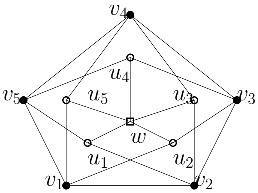
Since $\omega(M_k) = 2$, and M_k is a subgraph of M_{k+1}, it is clear that $\omega(M_{k+1}) \geq 2$.

WTS M_{k+1} is triangle-free. Suppose otherwise, and let T be a triangle in M_{k+1}. Since U is a stable set of G, we see that $|T \cap U| \leq 1$. Since $N_{M_{k+1}}(w) = U$, and since U is a stable set, we further see that $w \notin T$. Finally, since $M_{k+1}[V] = M_k$, and since M_k is triangle-free (by the induction hypothesis), we see that $T \not\subseteq V$.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof (continued). WTS $\omega(M_{k+1}) = 2$.



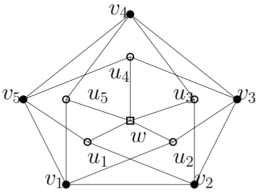
Since $\omega(M_k) = 2$, and M_k is a subgraph of M_{k+1}, it is clear that $\omega(M_{k+1}) \geq 2$.

WTS M_{k+1} is triangle-free. Suppose otherwise, and let T be a triangle in M_{k+1}. Since U is a stable set of G, we see that $|T \cap U| \leq 1$. Since $N_{M_{k+1}}(w) = U$, and since U is a stable set, we further see that $w \notin T$. Finally, since $M_{k+1}[V] = M_k$, and since M_k is triangle-free (by the induction hypothesis), we see that $T \not\subseteq V$. It now follows that $|T \cap U| = 1$ and $|T \cap V| = 2$.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

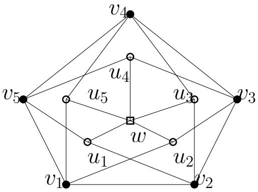
Proof (continued). Reminder: T be a triangle in M_{k+1}; $|T \cap U| = 1$ and $|T \cap V| = 2$.



Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof (continued). Reminder: T be a triangle in M_{k+1};
$|T \cap U| = 1$ and $|T \cap V| = 2$.

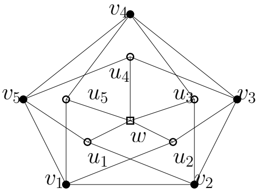


Let $p, q, r \in \{1, \ldots, k\}$ (with $q \neq r$) be s.t. $T = \{u_p, v_q, v_r\}$.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof (continued). Reminder: T be a triangle in M_{k+1};
$|T \cap U| = 1$ and $|T \cap V| = 2$.

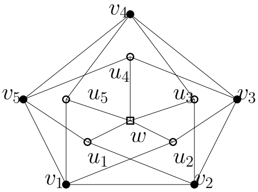


Let $p, q, r \in \{1, \ldots, k\}$ (with $q \neq r$) be s.t. $T = \{u_p, v_q, v_r\}$. By the construction of M_{k+1}, $u_p v_p \notin E(M_{k+1})$;

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof (continued). Reminder: T be a triangle in M_{k+1}; $|T \cap U| = 1$ and $|T \cap V| = 2$.

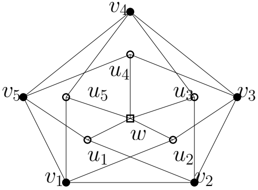


Let $p, q, r \in \{1, \dots, k\}$ (with $q \neq r$) be s.t. $T = \{u_p, v_q, v_r\}$. By the construction of M_{k+1}, $u_p v_p \notin E(M_{k+1})$; since T is a triangle, it follows that $p \notin \{q, r\}$. Since $u_p v_q \in E(M_{k+1})$, it follows from the construction of M_{k+1} that $v_p v_q \in E(M_k)$;

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof (continued). Reminder: T be a triangle in M_{k+1};
$|T \cap U| = 1$ and $|T \cap V| = 2$.

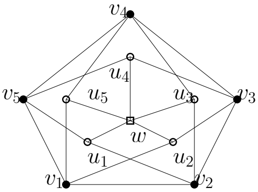


Let $p, q, r \in \{1, \ldots, k\}$ (with $q \neq r$) be s.t. $T = \{u_p, v_q, v_r\}$. By the construction of M_{k+1}, $u_p v_p \notin E(M_{k+1})$; since T is a triangle, it follows that $p \notin \{q, r\}$. Since $u_p v_q \in E(M_{k+1})$, it follows from the construction of M_{k+1} that $v_p v_q \in E(M_k)$; similarly, $v_p v_r \in E(M_k)$.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof (continued). Reminder: T be a triangle in M_{k+1}; $|T \cap U| = 1$ and $|T \cap V| = 2$.

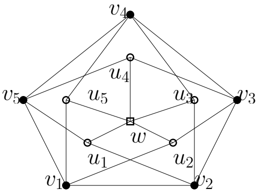


Let $p, q, r \in \{1, \dots, k\}$ (with $q \neq r$) be s.t. $T = \{u_p, v_q, v_r\}$. By the construction of M_{k+1}, $u_p v_p \notin E(M_{k+1})$; since T is a triangle, it follows that $p \notin \{q, r\}$. Since $u_p v_q \in E(M_{k+1})$, it follows from the construction of M_{k+1} that $v_p v_q \in E(M_k)$; similarly, $v_p v_r \in E(M_k)$. But now $\{v_p, v_q, v_r\}$ is a triangle in M_k, a contradiction. So, M_{k+1} is triangle-free, and we deduce that $\omega(M_{k+1}) = 2$.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

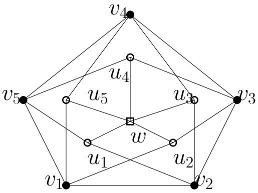
Proof (continued). WTS $\chi(M_{k+1}) = k + 1$.



Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof (continued). WTS $\chi(M_{k+1}) = k + 1$.

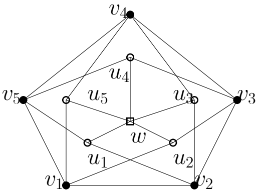


To see that $\chi(M_{k+1}) \leq k + 1$, properly color M_k with colors $1, \ldots, k$ (possible by the induction hypothesis), then color each u_i with the same color as v_i, and finally color w with color $k + 1$.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof (continued). WTS $\chi(M_{k+1}) = k + 1$.



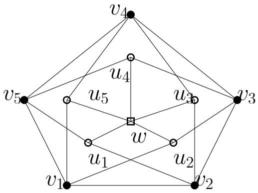
To see that $\chi(M_{k+1}) \leq k + 1$, properly color M_k with colors $1, \ldots, k$ (possible by the induction hypothesis), then color each u_i with the same color as v_i, and finally color w with color $k + 1$.

Suppose now $\chi(M_{k+1}) \leq k$, and fix a proper coloring $c : V(M_{k+1}) \to \{1, \ldots, k\}$ of M_{k+1}.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof (continued). WTS $\chi(M_{k+1}) = k + 1$.



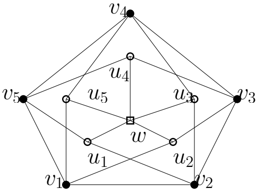
To see that $\chi(M_{k+1}) \leq k + 1$, properly color M_k with colors $1, \ldots, k$ (possible by the induction hypothesis), then color each u_i with the same color as v_i, and finally color w with color $k + 1$.

Suppose now $\chi(M_{k+1}) \leq k$, and fix a proper coloring $c : V(M_{k+1}) \to \{1, \ldots, k\}$ of M_{k+1}. We will construct a proper coloring of M_k with only $k - 1$ colors, contrary to the induction hypothesis.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

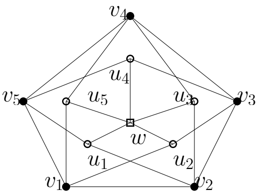
Proof (continued). WMA $c(w) = k$. Then no u_i receives color k.



Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof (continued). WMA $c(w) = k$. Then no u_i receives color k.

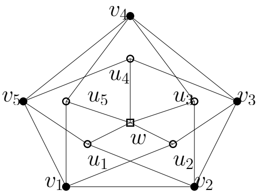


Let $V_k := \{v_i \mid c(v_i) = k\}$. Then V_k is a stable set.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof (continued). WMA $c(w) = k$. Then no u_i receives color k.

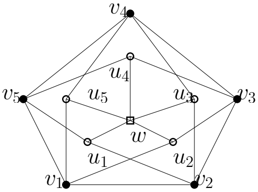


Let $V_k := \{v_i \mid c(v_i) = k\}$. Then V_k is a stable set. Now recolor each $v_i \in V_k$ with color $c(u_i)$.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Proof (continued). WMA $c(w) = k$. Then no u_i receives color k.



Let $V_k := \{v_i \mid c(v_i) = k\}$. Then V_k is a stable set. Now recolor each $v_i \in V_k$ with color $c(u_i)$. This produces a coloring of M_k that uses only $k - 1$ colors, a contradiction.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Theorem 1.2

There exist triangle-free graphs of arbitrarily large chromatic number. More precisely, for every positive integer k, there exists a graph G s.t. $\omega(G) = 2$ and $\chi(G) \geq k$.

Proof. This follows from Lemma 1.2.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Theorem 1.2

There exist triangle-free graphs of arbitrarily large chromatic number. More precisely, for every positive integer k, there exists a graph G s.t. $\omega(G) = 2$ and $\chi(G) \geq k$.

Proof. This follows from Lemma 1.2.

- Erdős (1961) applied the probabilistic method to demonstrate the existence of graphs with arbitrarily high girth and chromatic number.
  - The *girth* of a graph G that has at least one cycle is the length of the shortest cycle in G.

Lemma 1.1

For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.

Theorem 1.2

There exist triangle-free graphs of arbitrarily large chromatic number. More precisely, for every positive integer k, there exists a graph G s.t. $\omega(G) = 2$ and $\chi(G) \geq k$.

Proof. This follows from Lemma 1.2.

- Erdős (1961) applied the probabilistic method to demonstrate the existence of graphs with arbitrarily high girth and chromatic number.
  - The *girth* of a graph G that has at least one cycle is the length of the shortest cycle in G.
- Graphs of high girth are triangle-free, and so this result of Erdős is stronger than Theorem 1.2.

Part II: A very brief introduction to perfect graphs

- We saw that there exist graphs of small clique number, but large chromatic number.

Part II: A very brief introduction to perfect graphs

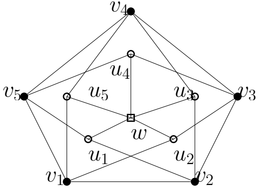
- We saw that there exist graphs of small clique number, but large chromatic number.
- At the other extreme, we might consider graphs for which $\chi = \omega$.

Part II: A very brief introduction to perfect graphs

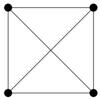
- We saw that there exist graphs of small clique number, but large chromatic number.
- At the other extreme, we might consider graphs for which $\chi = \omega$.
- This, however, turns out not to be a very interesting question.

Part II: A very brief introduction to perfect graphs

- We saw that there exist graphs of small clique number, but large chromatic number.
- At the other extreme, we might consider graphs for which $\chi = \omega$.
- This, however, turns out not to be a very interesting question.
- Indeed, suppose H is any graph at all, and let G be the disjoint union of H and $K_{\chi(H)}$.



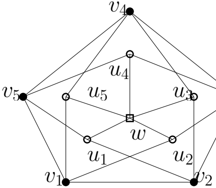
$H = M_4$



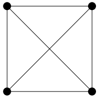
$K_{\chi(H)} = K_4$

Part II: A very brief introduction to perfect graphs

- We saw that there exist graphs of small clique number, but large chromatic number.
- At the other extreme, we might consider graphs for which $\chi = \omega$.
- This, however, turns out not to be a very interesting question.
- Indeed, suppose H is any graph at all, and let G be the disjoint union of H and $K_{\chi(H)}$.



$H = M_4$



$K_{\chi(H)} = K_4$

- Then $\chi(G) = \omega(G)$, but we can say very little about the structure of G (since G was built starting from an arbitrary graph H).

Definition

A graph is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$.

Definition

A graph is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$.

- Since every graph is an induced subgraph of itself, we see that every perfect graph G satisfies $\chi(G) = \omega(G)$.

Definition

A graph is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$.

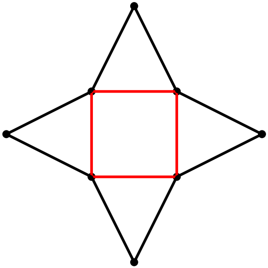
- Since every graph is an induced subgraph of itself, we see that every perfect graph G satisfies $\chi(G) = \omega(G)$.
- Importantly, though, in a perfect graph, $\chi = \omega$ should hold not only for the graph itself, but also for all its induced subgraphs.

Part III: Chordal graphs

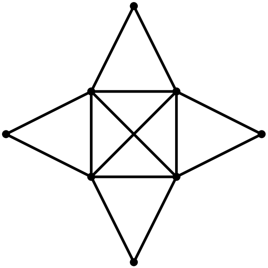
Definition

A graph is *chordal* (or *triangulated*) if every cycle of length strictly greater than three has a chord (a *chord* of a cycle is an edge joining two nonconsecutive vertices of the cycle).

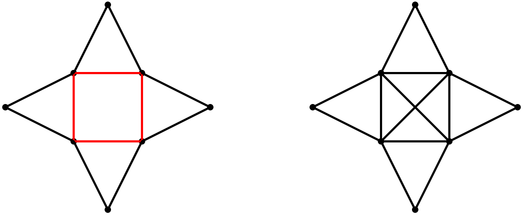
- In other words, a graph is *chordal* if it contains no induced cycles of length at least four.



not chordal

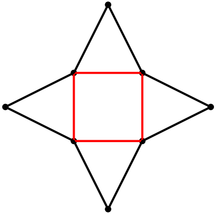


chordal

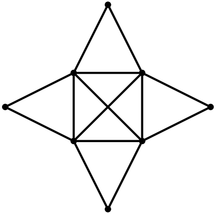


not chordal   chordal

- Note that all induced subgraphs of a chordal graph are chordal.

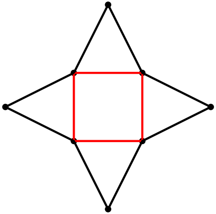


not chordal

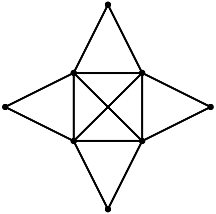


chordal

- Note that all induced subgraphs of a chordal graph are chordal.
- Chordal graphs were one of the first classes of graphs to be recognized as perfect; the study of chordal graphs can be seen as the beginning of the theory of perfect graphs.



not chordal

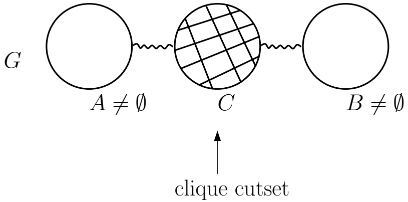


chordal

- Note that all induced subgraphs of a chordal graph are chordal.
- Chordal graphs were one of the first classes of graphs to be recognized as perfect; the study of chordal graphs can be seen as the beginning of the theory of perfect graphs.
- As we shall see, there are efficient algorithms for recognizing chordal graphs and for solving the vertex coloring and related optimization problems on chordal graphs.

Definition

A *cutset* of a graph is a set of vertices whose deletion yields a disconnected graph. More precisely, a *cutset* of a graph G is a (possibly empty) set $S \subsetneq V(G)$ s.t. $G \setminus S$ is disconnected. A *clique-cutset* is a cutset that is a clique, that is, a *clique-cutset* of a graph G is a clique $C \subsetneq V(G)$ of G s.t. $G \setminus C$ is disconnected.



Lemma 3.1

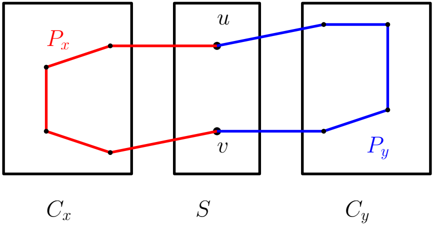
Let G be a chordal graph that is not complete, let x and y be non-adjacent vertices of G, and let S be a minimal cutset of G separating x and y. Then S is a clique of G.

Proof (outline).

Lemma 3.1

Let G be a chordal graph that is not complete, let x and y be non-adjacent vertices of G, and let S be a minimal cutset of G separating x and y. Then S is a clique of G.

Proof (outline). Suppose that S is not a clique, and let u and v be two nonadjacent vertices of S. Let C_x be the component of $G \setminus S$ that contains x, and let C_y be the component of $G \setminus S$ that contains y.



Now $P_x \cup P_y$ is an induced cycle of length at least four in G, a contradiction.

Lemma 3.1

Let G be a chordal graph that is not complete, let x and y be non-adjacent vertices of G, and let S be a minimal cutset of G separating x and y. Then S is a clique of G.

Lemma 3.1

Let G be a chordal graph that is not complete, let x and y be non-adjacent vertices of G, and let S be a minimal cutset of G separating x and y. Then S is a clique of G.

Theorem 3.2

If G is a chordal graph, then either G is a complete graph or G admits a clique-cutset.

Proof.

Lemma 3.1

Let G be a chordal graph that is not complete, let x and y be non-adjacent vertices of G, and let S be a minimal cutset of G separating x and y. Then S is a clique of G.

Theorem 3.2

If G is a chordal graph, then either G is a complete graph or G admits a clique-cutset.

Proof. Let G be a chordal graph that is not complete. Let x and y be non-adjacent vertices of G, and let S be a minimal cutset of G separating x from y. By Lemma 3.1, S is a clique. It follows that S is a clique-cutset of G.

Definition

A graph is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$.

Corollary 3.3

Chordal graphs are perfect.

Proof.

Definition

A graph is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$.

Corollary 3.3

Chordal graphs are perfect.

Proof. Since every induced subgraph of a chordal graph is chordal, it is enough to show that every chordal graph G satisfies $\chi(G) = \omega(G)$.

Definition

A graph is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$.

Corollary 3.3

Chordal graphs are perfect.

Proof. Since every induced subgraph of a chordal graph is chordal, it is enough to show that every chordal graph G satisfies $\chi(G) = \omega(G)$. So, fix a chordal graph G, and assume inductively that all chordal graphs G' on fewer than $|V(G)|$ vertices satisfy $\chi(G') = \omega(G')$. WTS $\chi(G) = \omega(G)$.

Definition

A graph is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$.

Corollary 3.3

Chordal graphs are perfect.

Proof. Since every induced subgraph of a chordal graph is chordal, it is enough to show that every chordal graph G satisfies $\chi(G) = \omega(G)$. So, fix a chordal graph G, and assume inductively that all chordal graphs G' on fewer than $|V(G)|$ vertices satisfy $\chi(G') = \omega(G')$. WTS $\chi(G) = \omega(G)$. If G is a complete graph, then it is clear that $\chi(G) = \omega(G)$.

Definition

A graph is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$.

Corollary 3.3

Chordal graphs are perfect.

Proof. Since every induced subgraph of a chordal graph is chordal, it is enough to show that every chordal graph G satisfies $\chi(G) = \omega(G)$. So, fix a chordal graph G, and assume inductively that all chordal graphs G' on fewer than $|V(G)|$ vertices satisfy $\chi(G') = \omega(G')$. WTS $\chi(G) = \omega(G)$. If G is a complete graph, then it is clear that $\chi(G) = \omega(G)$. So, assume that G is not complete. Then by Theorem 3.2, G admits a clique-cutset, call it C.

Corollary 3.3

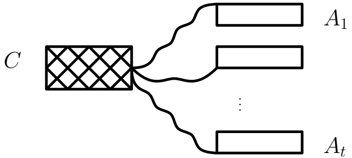
Chordal graphs are perfect.

Proof (continued).

Corollary 3.3

Chordal graphs are perfect.

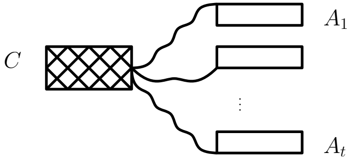
Proof (continued). Let $A_1, \ldots, A_t$ ($t \geq 2$) be the vertex sets of the components of $G \setminus C$. For all $i \in \{1, \ldots, t\}$, let $G_i := G[A_i \cup C]$.



Corollary 3.3

Chordal graphs are perfect.

Proof (continued). Let $A_1, \ldots, A_t$ ($t \geq 2$) be the vertex sets of the components of $G \setminus C$. For all $i \in \{1, \ldots, t\}$, let $G_i := G[A_i \cup C]$.

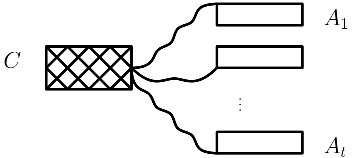


Note that every clique of G is in fact a clique of one of $G_1, \ldots, G_t$, and it follows that $\omega(G) = \max\{\omega(G_1), \ldots, \omega(G_t)\}$.

Corollary 3.3

Chordal graphs are perfect.

Proof (continued). Let $A_1, \dots, A_t$ ($t \geq 2$) be the vertex sets of the components of $G \setminus C$. For all $i \in \{1, \dots, t\}$, let $G_i := G[A_i \cup C]$.

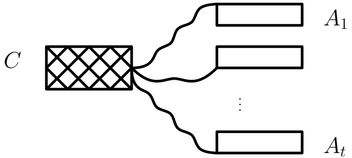


Note that every clique of G is in fact a clique of one of $G_1, \dots, G_t$, and it follows that $\omega(G) = \max\{\omega(G_1), \dots, \omega(G_t)\}$. On the other hand, by Lemma 2.1 from Lecture Notes 4, we have that $\chi(G) = \max\{\chi(G_1), \dots, \chi(G_t)\}$.

Corollary 3.3

Chordal graphs are perfect.

Proof (continued). Let $A_1, \dots, A_t$ ($t \geq 2$) be the vertex sets of the components of $G \setminus C$. For all $i \in \{1, \dots, t\}$, let $G_i := G[A_i \cup C]$.

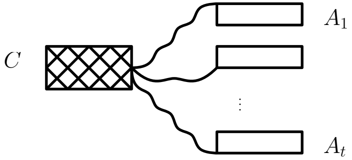


Note that every clique of G is in fact a clique of one of $G_1, \dots, G_t$, and it follows that $\omega(G) = \max\{\omega(G_1), \dots, \omega(G_t)\}$. On the other hand, by Lemma 2.1 from Lecture Notes 4, we have that $\chi(G) = \max\{\chi(G_1), \dots, \chi(G_t)\}$. Finally, for all $i \in \{1, \dots, t\}$, the induction hypothesis guarantees that $\chi(G_i) = \omega(G_i)$.

Corollary 3.3

Chordal graphs are perfect.

Proof (continued).



So,

$$\begin{aligned}\chi(G) &= \max\{\chi(G_1), \ldots, \chi(G_t)\} \\ &= \max\{\omega(G_1), \ldots, \omega(G_t)\} \\ &= \omega(G),\end{aligned}$$

which is what we needed.

Definition

A vertex x of a graph G is *simplicial* if $N_G(x)$ is a clique of G.

Definition

A vertex x of a graph G is *simplicial* if $N_G(x)$ is a clique of G.

Theorem 3.4 [Dirac, 1961]

Every chordal graph has a simplicial vertex. Moreover, every chordal graph that is not complete has (at least) two non-adjacent simplicial vertices.

Proof (outline).

Definition

A vertex x of a graph G is *simplicial* if $N_G(x)$ is a clique of G.

Theorem 3.4 [Dirac, 1961]

Every chordal graph has a simplicial vertex. Moreover, every chordal graph that is not complete has (at least) two non-adjacent simplicial vertices.

Proof (outline). By induction on the number of vertices.

Definition

A vertex x of a graph G is *simplicial* if $N_G(x)$ is a clique of G.

Theorem 3.4 [Dirac, 1961]

Every chordal graph has a simplicial vertex. Moreover, every chordal graph that is not complete has (at least) two non-adjacent simplicial vertices.

Proof (outline). By induction on the number of vertices. Obviously true for complete graphs.

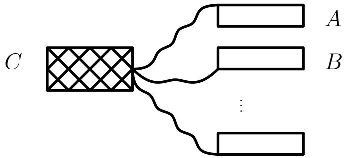
Definition

A vertex x of a graph G is *simplicial* if $N_G(x)$ is a clique of G.

Theorem 3.4 [Dirac, 1961]

Every chordal graph has a simplicial vertex. Moreover, every chordal graph that is not complete has (at least) two non-adjacent simplicial vertices.

Proof (outline). By induction on the number of vertices. Obviously true for complete graphs. If G is chordal, but not complete, then by induction hypothesis, $G[A \cup C]$ has a simplicial vertex $a \in A$, and $G[B \cup C]$ has a simplicial vertex $b \in B$. Now a and b are non-adjacent simplicial vertices of G.



Definition

A vertex x of a graph G is *simplicial* if $N_G(x)$ is a clique of G.

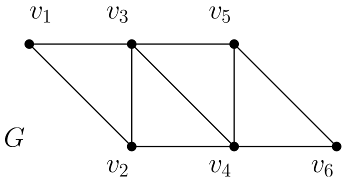
Definition

A vertex x of a graph G is *simplicial* if $N_G(x)$ is a clique of G.

Definition

A *simplicial elimination ordering* (sometimes also called a *perfect elimination ordering*) of a graph G is an ordering $v_1, \dots, v_n$ of its vertices s.t. for all $i \in \{1, \dots, n\}$, v_i is simplicial in the graph $G[v_i, \dots, v_n]$.

- For instance, $v_1, \dots, v_6$ is a simplicial elimination ordering of the graph G in the picture below.



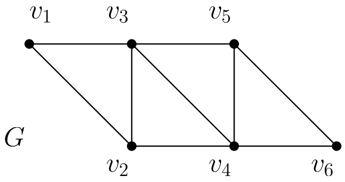
Definition

A vertex x of a graph G is *simplicial* if $N_G(x)$ is a clique of G.

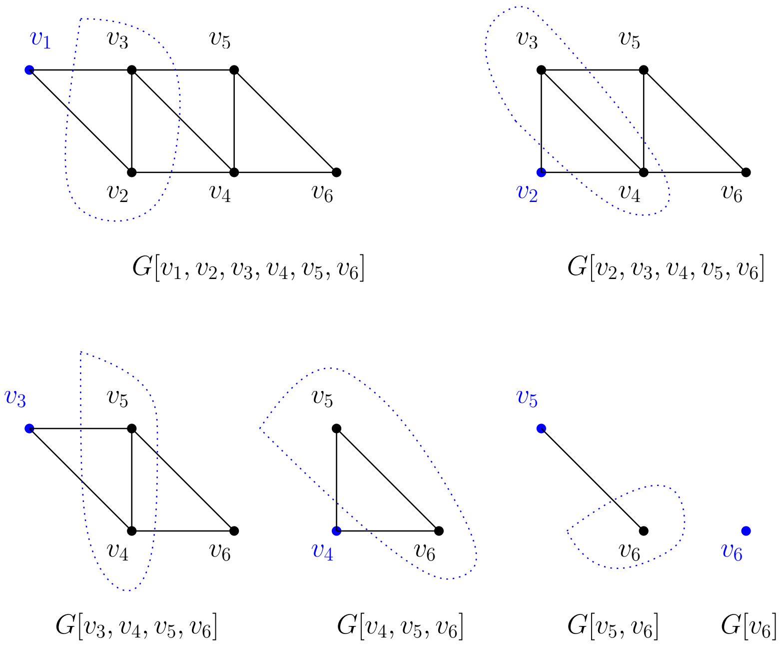
Definition

A *simplicial elimination ordering* (sometimes also called a *perfect elimination ordering*) of a graph G is an ordering $v_1, \dots, v_n$ of its vertices s.t. for all $i \in \{1, \dots, n\}$, v_i is simplicial in the graph $G[v_i, \dots, v_n]$.

- For instance, $v_1, \dots, v_6$ is a simplicial elimination ordering of the graph G in the picture below.



- Indeed (next slide).



v_1 v_3 v_5 v_2 v_4 v_6
$G[v_1, v_2, v_3, v_4, v_5, v_6]$
v_3 v_5 v_2 v_4 v_6
$G[v_2, v_3, v_4, v_5, v_6]$
v_3 v_5 v_4 v_6
$G[v_3, v_4, v_5, v_6]$
v_5 v_4 v_6
$G[v_4, v_5, v_6]$
v_5 v_6
$G[v_5, v_6]$
v_6
$G[v_6]$

Theorem 3.5 [Fulkerson and Gross, 1965]

For a graph G, the following statements are equivalent:

(i) G is chordal;

(ii) G has a simplicial elimination ordering;

(iii) for all non-adjacent vertices x and y of G, every minimal cutset of G separating x from y is a clique.

Proof (outline).

Theorem 3.5 [Fulkerson and Gross, 1965]

For a graph G, the following statements are equivalent:

(i) G is chordal;

(ii) G has a simplicial elimination ordering;

(iii) for all non-adjacent vertices x and y of G, every minimal cutset of G separating x from y is a clique.

Proof (outline). **(i)** $\Rightarrow$ **(iii):** This follows from Lemma 3.1.

Theorem 3.5 [Fulkerson and Gross, 1965]

For a graph G, the following statements are equivalent:

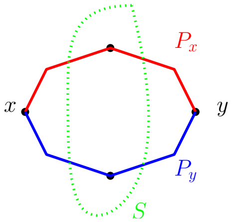
(i) G is chordal;

(ii) G has a simplicial elimination ordering;

(iii) for all non-adjacent vertices x and y of G, every minimal cutset of G separating x from y is a clique.

Proof (outline). **(i) ⇒ (iii):** This follows from Lemma 3.1.

(iii) ⇒ (i): If (i) is false, then (iii) is false. (Details: Lecture Notes.)



Theorem 3.5 [Fulkerson and Gross, 1965]

For a graph G, the following statements are equivalent:

(i) G is chordal;

(ii) G has a simplicial elimination ordering;

(iii) for all non-adjacent vertices x and y of G, every minimal cutset of G separating x from y is a clique.

Proof (outline, continued). **(i)** $\Rightarrow$ **(ii):**

Theorem 3.5 [Fulkerson and Gross, 1965]

For a graph G, the following statements are equivalent:

(i) G is chordal;

(ii) G has a simplicial elimination ordering;

(iii) for all non-adjacent vertices x and y of G, every minimal cutset of G separating x from y is a clique.

Proof (outline, continued). **(i) $\Rightarrow$ (ii):** We proceed by induction on the number of vertices.

Theorem 3.5 [Fulkerson and Gross, 1965]

For a graph G, the following statements are equivalent:

(i) G is chordal;

(ii) G has a simplicial elimination ordering;

(iii) for all non-adjacent vertices x and y of G, every minimal cutset of G separating x from y is a clique.

Proof (outline, continued). **(i) $\Rightarrow$ (ii):** We proceed by induction on the number of vertices. Clearly, the claim holds for one-vertex graphs.

Theorem 3.5 [Fulkerson and Gross, 1965]

For a graph G, the following statements are equivalent:

(i) G is chordal;

(ii) G has a simplicial elimination ordering;

(iii) for all non-adjacent vertices x and y of G, every minimal cutset of G separating x from y is a clique.

Proof (outline, continued). **(i)** $\Rightarrow$ **(ii):** We proceed by induction on the number of vertices. Clearly, the claim holds for one-vertex graphs. Now, fix a positive integer n, and assume that the claim holds for all chordal graphs on n vertices. Let H be a chordal graph on $n+1$ vertices. By Theorem 3.4, H has at least one simplicial vertex, call it x_0.

Theorem 3.5 [Fulkerson and Gross, 1965]

For a graph G, the following statements are equivalent:

(i) G is chordal;

(ii) G has a simplicial elimination ordering;

(iii) for all non-adjacent vertices x and y of G, every minimal cutset of G separating x from y is a clique.

Proof (outline, continued). **(i) $\Rightarrow$ (ii):** We proceed by induction on the number of vertices. Clearly, the claim holds for one-vertex graphs. Now, fix a positive integer n, and assume that the claim holds for all chordal graphs on n vertices. Let H be a chordal graph on $n+1$ vertices. By Theorem 3.4, H has at least one simplicial vertex, call it x_0. Then $H \setminus x_0$ is a chordal graph on n vertices, and so by the induction hypothesis, $H \setminus x_0$ has a simplicial elimination ordering, say $x_1, \dots, x_n$.

Theorem 3.5 [Fulkerson and Gross, 1965]

For a graph G, the following statements are equivalent:

(i) G is chordal;

(ii) G has a simplicial elimination ordering;

(iii) for all non-adjacent vertices x and y of G, every minimal cutset of G separating x from y is a clique.

Proof (outline, continued). **(i) $\Rightarrow$ (ii):** We proceed by induction on the number of vertices. Clearly, the claim holds for one-vertex graphs. Now, fix a positive integer n, and assume that the claim holds for all chordal graphs on n vertices. Let H be a chordal graph on $n+1$ vertices. By Theorem 3.4, H has at least one simplicial vertex, call it x_0. Then $H \setminus x_0$ is a chordal graph on n vertices, and so by the induction hypothesis, $H \setminus x_0$ has a simplicial elimination ordering, say $x_1, \dots, x_n$. But now $x_0, x_1, \dots, x_n$ is a simplicial elimination ordering of H.

Theorem 3.5 [Fulkerson and Gross, 1965]

For a graph G, the following statements are equivalent:

(i) G is chordal;

(ii) G has a simplicial elimination ordering;

(iii) for all non-adjacent vertices x and y of G, every minimal cutset of G separating x from y is a clique.

Proof (outline, continued). **(ii)** $\Rightarrow$ **(i):**

Theorem 3.5 [Fulkerson and Gross, 1965]

For a graph G, the following statements are equivalent:

(i) G is chordal;

(ii) G has a simplicial elimination ordering;

(iii) for all non-adjacent vertices x and y of G, every minimal cutset of G separating x from y is a clique.

Proof (outline, continued). **(ii) $\Rightarrow$ (i):** Suppose that $v_1, \dots, v_n$ is a simplicial elimination ordering of G; we claim that G is chordal.

Theorem 3.5 [Fulkerson and Gross, 1965]

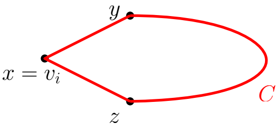
For a graph G, the following statements are equivalent:

(i) G is chordal;

(ii) G has a simplicial elimination ordering;

(iii) for all non-adjacent vertices x and y of G, every minimal cutset of G separating x from y is a clique.

Proof (outline, continued). **(ii) $\Rightarrow$ (i):** Suppose that $v_1, \dots, v_n$ is a simplicial elimination ordering of G; we claim that G is chordal. Let C be an induced cycle of G. Let $x = v_i$ be the lowest-indexed vertex that belongs C, and let y, z be the two neighbors of x in C.



Then $yz \in E(G)$, and so C is a triangle. Thus, G is chordal.

Theorem 3.5 [Fulkerson and Gross, 1965]

For a graph G, the following statements are equivalent:

(i) G is chordal;

(ii) G has a simplicial elimination ordering;

(iii) for all non-adjacent vertices x and y of G, every minimal cutset of G separating x from y is a clique.

Theorem 3.5 [Fulkerson and Gross, 1965]

For a graph G, the following statements are equivalent:

(i) G is chordal;

(ii) G has a simplicial elimination ordering;

(iii) for all non-adjacent vertices x and y of G, every minimal cutset of G separating x from y is a clique.

- Note that Theorem 3.5 gives an $O(n^4)$ time recognition algorithm for chordal graphs (we repeatedly search for simplicial vertices).

Theorem 3.5 [Fulkerson and Gross, 1965]

For a graph G, the following statements are equivalent:

(i) G is chordal;

(ii) G has a simplicial elimination ordering;

(iii) for all non-adjacent vertices x and y of G, every minimal cutset of G separating x from y is a clique.

- Note that Theorem 3.5 gives an $O(n^4)$ time recognition algorithm for chordal graphs (we repeatedly search for simplicial vertices).
- In fact, chordal graphs can be recognized in $O(n + m)$ time using the so called Lexicographic breadth-first-search (LexBFS) due to Rose, Tarjan, and Lueker (1976), but we omit the details.

- Our next goal is to construct efficient algorithms solving the Graph Coloring, Maximum Clique, Maximum Stable Set, and Minimum Clique Cover problems on chordal graphs.

- Our next goal is to construct efficient algorithms solving the GRAPH COLORING, MAXIMUM CLIQUE, MAXIMUM STABLE SET, and MINIMUM CLIQUE COVER problems on chordal graphs.
- For the rest of this lecture, G is a chordal graph on n vertices, and $v_1, \dots, v_n$ is a simplicial elimination ordering on G.

- Our next goal is to construct efficient algorithms solving the GRAPH COLORING, MAXIMUM CLIQUE, MAXIMUM STABLE SET, and MINIMUM CLIQUE COVER problems on chordal graphs.
- For the rest of this lecture, G is a chordal graph on n vertices, and $v_1, \dots, v_n$ is a simplicial elimination ordering on G.
- For each $i \in \{1, \dots, n\}$, set $X_i := N_G[v_i] \cap \{v_i, \dots, v_n\}$.
  - So, X_i is the closed neighborhood of v_i in the graph $G[v_i, \dots, v_n]$.

Lemma 3.6

$X_1, \ldots, X_n$ are all cliques of G. Furthermore, for every maximal clique C of G, there exists some $i \in \{1, \ldots, n\}$ s.t. $C = X_i$.[a]

[a] However, not all X_i's need be maximal cliques.

Proof.

Lemma 3.6

$X_1, \ldots, X_n$ are all cliques of G. Furthermore, for every maximal clique C of G, there exists some $i \in \{1, \ldots, n\}$ s.t. $C = X_i$.[a]

[a] However, not all X_i's need be maximal cliques.

Proof. The fact that the sets X_i are cliques follows immediately from the definition of a simplicial elimination ordering and the construction of the sets X_i.

Lemma 3.6

$X_1, \ldots, X_n$ are all cliques of G. Furthermore, for every maximal clique C of G, there exists some $i \in \{1, \ldots, n\}$ s.t. $C = X_i$.[a]

[a] However, not all X_i's need be maximal cliques.

Proof. The fact that the sets X_i are cliques follows immediately from the definition of a simplicial elimination ordering and the construction of the sets X_i. Now, let C be a maximal clique of G. Let $i \in \{1, \ldots, n\}$ be minimal with $v_i \in C$.

Lemma 3.6

$X_1, \ldots, X_n$ are all cliques of G. Furthermore, for every maximal clique C of G, there exists some $i \in \{1, \ldots, n\}$ s.t. $C = X_i$.[a]

[a] However, not all X_i's need be maximal cliques.

Proof. The fact that the sets X_i are cliques follows immediately from the definition of a simplicial elimination ordering and the construction of the sets X_i. Now, let C be a maximal clique of G. Let $i \in \{1, \ldots, n\}$ be minimal with $v_i \in C$. Then clearly, $C \subseteq X_i$. Since C is a maximal clique, and X_i is a clique, it follows that $C = X_i$.

Lemma 3.6

$X_1, \ldots, X_n$ are all cliques of G. Furthermore, for every maximal clique C of G, there exists some $i \in \{1, \ldots, n\}$ s.t. $C = X_i$.[a]

[a] However, not all X_i's need be maximal cliques.

Proof. The fact that the sets X_i are cliques follows immediately from the definition of a simplicial elimination ordering and the construction of the sets X_i. Now, let C be a maximal clique of G. Let $i \in \{1, \ldots, n\}$ be minimal with $v_i \in C$. Then clearly, $C \subseteq X_i$. Since C is a maximal clique, and X_i is a clique, it follows that $C = X_i$.

Theorem 3.7 [Fulkerson and Gross, 1965]

G has at most n maximal cliques. Furthermore, equality holds iff G is edgeless.

Proof. This follows from Lemma 3.6 (Details: Lecture Notes.)

Definition

A *clique cover* of a graph H is a partition of $V(H)$ into cliques. The *clique cover number* of H, denoted by $\overline{\chi}(H)$, is the smallest size of a clique cover of H; a *minimum clique cover* of H is a clique cover of size precisely $\overline{\chi}(H)$.

Definition

A *clique cover* of a graph H is a partition of $V(H)$ into cliques. The *clique cover number* of H, denoted by $\overline{\chi}(H)$, is the smallest size of a clique cover of H; a *minimum clique cover* of H is a clique cover of size precisely $\overline{\chi}(H)$.

- Obviously, every graph H satisfies $\alpha(H) \leq \overline{\chi}(H)$.

Definition

A *clique cover* of a graph H is a partition of $V(H)$ into cliques. The *clique cover number* of H, denoted by $\overline{\chi}(H)$, is the smallest size of a clique cover of H; a *minimum clique cover* of H is a clique cover of size precisely $\overline{\chi}(H)$.

- Obviously, every graph H satisfies $\alpha(H) \leq \overline{\chi}(H)$.
- Since proper colorings correspond to partitions of the vertex set into stable sets (color classes), every graph H satisfies $\overline{\chi}(H) = \chi(\overline{H})$.

- We define a (finite) sequence $i_1, \ldots, i_t$ as follows.

- We define a (finite) sequence $i_1, \ldots, i_t$ as follows.
- First, let $i_1 := 1$.

- We define a (finite) sequence $i_1, \ldots, i_t$ as follows.
- First, let $i_1 := 1$.
- Once $i_1, \ldots, i_{j-1}$ have been defined, we either terminate or extend the sequence, as follows.

- We define a (finite) sequence $i_1, \dots, i_t$ as follows.
- First, let $i_1 := 1$.
- Once $i_1, \dots, i_{j-1}$ have been defined, we either terminate or extend the sequence, as follows.
- If $V(G) = X_{i_1} \cup \dots \cup X_{i_{j-1}}$, then we set $t = j - 1$, and we terminate the sequence; otherwise, we let $i_j \in \{1, \dots, n\}$ be the smallest index s.t. $v_{i_j} \notin X_{i_1} \cup \dots \cup X_{i_{j-1}}$.

- We define a (finite) sequence $i_1, \dots, i_t$ as follows.
- First, let $i_1 := 1$.
- Once $i_1, \dots, i_{j-1}$ have been defined, we either terminate or extend the sequence, as follows.
- If $V(G) = X_{i_1} \cup \dots \cup X_{i_{j-1}}$, then we set $t = j - 1$, and we terminate the sequence; otherwise, we let $i_j \in \{1, \dots, n\}$ be the smallest index s.t. $v_{i_j} \notin X_{i_1} \cup \dots \cup X_{i_{j-1}}$.
- Set $Y_1 := X_{i_1}$, and for all $j \in \{2, \dots, t\}$, set $Y_j := X_{i_j} \setminus (Y_1 \cup \dots \cup Y_{j-1})$.

- We define a (finite) sequence $i_1, \dots, i_t$ as follows.
- First, let $i_1 := 1$.
- Once $i_1, \dots, i_{j-1}$ have been defined, we either terminate or extend the sequence, as follows.
- If $V(G) = X_{i_1} \cup \dots \cup X_{i_{j-1}}$, then we set $t = j - 1$, and we terminate the sequence; otherwise, we let $i_j \in \{1, \dots, n\}$ be the smallest index s.t. $v_{i_j} \notin X_{i_1} \cup \dots \cup X_{i_{j-1}}$.
- Set $Y_1 := X_{i_1}$, and for all $j \in \{2, \dots, t\}$, set $Y_j := X_{i_j} \setminus (Y_1 \cup \dots \cup Y_{j-1})$.

Theorem 3.8 [Gavril, 1972]

The set $\{v_{i_1}, \dots, v_{i_t}\}$ is a maximum stable set of G, and $(Y_1, \dots, Y_t)$ is a minimum clique cover of G.

Proof (outline).

- We define a (finite) sequence $i_1, \dots, i_t$ as follows.
- First, let $i_1 := 1$.
- Once $i_1, \dots, i_{j-1}$ have been defined, we either terminate or extend the sequence, as follows.
- If $V(G) = X_{i_1} \cup \dots \cup X_{i_{j-1}}$, then we set $t = j - 1$, and we terminate the sequence; otherwise, we let $i_j \in \{1, \dots, n\}$ be the smallest index s.t. $v_{i_j} \notin X_{i_1} \cup \dots \cup X_{i_{j-1}}$.
- Set $Y_1 := X_{i_1}$, and for all $j \in \{2, \dots, t\}$, set $Y_j := X_{i_j} \setminus (Y_1 \cup \dots \cup Y_{j-1})$.

Theorem 3.8 [Gavril, 1972]

The set $\{v_{i_1}, \dots, v_{i_t}\}$ is a maximum stable set of G, and $(Y_1, \dots, Y_t)$ is a minimum clique cover of G.

Proof (outline). The fact that $\{v_{i_1}, \dots, v_{i_t}\}$ is a stable set and $(Y_1, \dots, Y_t)$ is a clique-cover of G follows from the construction.

- We define a (finite) sequence $i_1, \dots, i_t$ as follows.
- First, let $i_1 := 1$.
- Once $i_1, \dots, i_{j-1}$ have been defined, we either terminate or extend the sequence, as follows.
- If $V(G) = X_{i_1} \cup \dots \cup X_{i_{j-1}}$, then we set $t = j - 1$, and we terminate the sequence; otherwise, we let $i_j \in \{1, \dots, n\}$ be the smallest index s.t. $v_{i_j} \notin X_{i_1} \cup \dots \cup X_{i_{j-1}}$.
- Set $Y_1 := X_{i_1}$, and for all $j \in \{2, \dots, t\}$, set $Y_j := X_{i_j} \setminus (Y_1 \cup \dots \cup Y_{j-1})$.

Theorem 3.8 [Gavril, 1972]

The set $\{v_{i_1}, \dots, v_{i_t}\}$ is a maximum stable set of G, and $(Y_1, \dots, Y_t)$ is a minimum clique cover of G.

Proof (outline). The fact that $\{v_{i_1}, \dots, v_{i_t}\}$ is a stable set and $(Y_1, \dots, Y_t)$ is a clique-cover of G follows from the construction. But now $t \leq \alpha(G) \leq \overline{\chi}(G) \leq t$, and so $\alpha(G) = \overline{\chi}(G) = t$. So, our stable set is maximum, and our clique cover is minimum.

Lemma 3.9

G can be optimally colored (i.e. properly colored using precisely $\chi(G)$ colors) by applying the greedy coloring algorithm to G with the ordering $v_n, \ldots, v_1$.[a]

[a] So, we are using the reverse of our simplicial elimination ordering.

Proof.

Lemma 3.9

G can be optimally colored (i.e. properly colored using precisely $\chi(G)$ colors) by applying the greedy coloring algorithm to G with the ordering $v_n, \ldots, v_1$.[a]

[a] So, we are using the reverse of our simplicial elimination ordering.

Proof. Clearly, the greedy coloring produces a proper coloring of G.

Lemma 3.9

G can be optimally colored (i.e. properly colored using precisely $\chi(G)$ colors) by applying the greedy coloring algorithm to G with the ordering $v_n, \ldots, v_1$.[a]

[a] So, we are using the reverse of our simplicial elimination ordering.

Proof. Clearly, the greedy coloring produces a proper coloring of G. If we apply the greedy coloring algorithm to G with the ordering $v_n, \ldots, v_1$, then when we reach a vertex v_i, the neighbors of v_i that have already been colored are precisely those from the clique $X_i \setminus \{v_i\}$,

Lemma 3.9

G can be optimally colored (i.e. properly colored using precisely $\chi(G)$ colors) by applying the greedy coloring algorithm to G with the ordering $v_n, \dots, v_1$.[a]

[a] So, we are using the reverse of our simplicial elimination ordering.

Proof. Clearly, the greedy coloring produces a proper coloring of G. If we apply the greedy coloring algorithm to G with the ordering $v_n, \dots, v_1$, then when we reach a vertex v_i, the neighbors of v_i that have already been colored are precisely those from the clique $X_i \setminus \{v_i\}$, and consequently, at most $\omega(G) - 1$ neighbors of v_i have already been colored. Thus, the greedy algorithm applied to G with this ordering uses no more than $\omega(G)$ colors.

Lemma 3.9

G can be optimally colored (i.e. properly colored using precisely $\chi(G)$ colors) by applying the greedy coloring algorithm to G with the ordering $v_n, \dots, v_1$.[a]

[a] So, we are using the reverse of our simplicial elimination ordering.

Proof. Clearly, the greedy coloring produces a proper coloring of G. If we apply the greedy coloring algorithm to G with the ordering $v_n, \dots, v_1$, then when we reach a vertex v_i, the neighbors of v_i that have already been colored are precisely those from the clique $X_i \setminus \{v_i\}$, and consequently, at most $\omega(G) - 1$ neighbors of v_i have already been colored. Thus, the greedy algorithm applied to G with this ordering uses no more than $\omega(G)$ colors. Since every graph H satisfies $\chi(H) \geq \omega(H)$, it follows that the greedy coloring algorithm used precisely $\omega(G)$ colors, and that the coloring that it produced is optimal.

Lemma 3.6

$X_1, \ldots, X_n$ are all cliques of G. Furthermore, for every maximal clique C of G, there exists some $i \in \{1, \ldots, n\}$ s.t. $C = X_i$.

Theorem 3.8 [Gavril, 1972]

The set $\{v_{i_1}, \ldots, v_{i_t}\}$ is a maximum stable set of G, and $(Y_1, \ldots, Y_t)$ is a minimum clique cover of G.

Lemma 3.9

G can be optimally colored (i.e. properly colored using precisely $\chi(G)$ colors) by applying the greedy coloring algorithm to G with the ordering $v_n, \ldots, v_1$.

Lemma 3.6

$X_1, \ldots, X_n$ are all cliques of G. Furthermore, for every maximal clique C of G, there exists some $i \in \{1, \ldots, n\}$ s.t. $C = X_i$.

Theorem 3.8 [Gavril, 1972]

The set $\{v_{i_1}, \ldots, v_{i_t}\}$ is a maximum stable set of G, and $(Y_1, \ldots, Y_t)$ is a minimum clique cover of G.

Lemma 3.9

G can be optimally colored (i.e. properly colored using precisely $\chi(G)$ colors) by applying the greedy coloring algorithm to G with the ordering $v_n, \ldots, v_1$.

- Clearly, Lemma 3.6, Theorem 3.8, and Lemma 3.9 yield polynomial time algorithms for finding a maximum clique, a maximum stable set, a minimum clique-cover, and an optimal coloring of a chordal graph.