NDMI012: Combinatorics and Graph Theory 2

Lecture #7
Chordal graphs

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1 Triangle-free graphs of arbitrarily large chromatic number

Clearly, every graph G satisfies $\omega(G) \leq \chi(G)$.[1] So, the simplest way to construct a graph of high chromatic number is to construct a graph that has a large clique number. However, as we shall see, it is possible to construct graphs of small clique number and large chromatic number.

A *triangle* in a graph G is a clique of size three. A graph is *triangle-free* if it contains no triangles. So, a graph is triangle-free if and only if its clique number is at most two. Our goal in this section is to construct a family of triangle-free graphs of arbitrarily large chromatic number. There are several known constructions; here, we give the one due to Mycielski (1955).

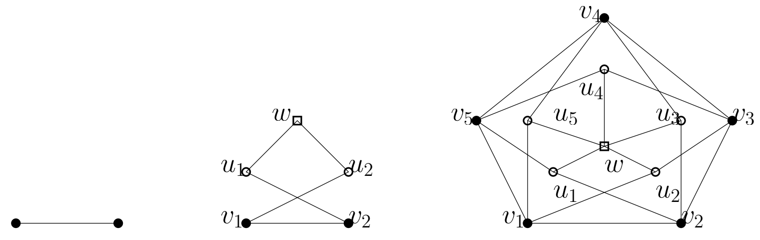
The Mycielski graphs $\{M_k\}_{k=2}^{\infty}$ are defined recursively, as follows. First, let $M_2 = K_2$. Next, fix an integer k, and suppose M_k has been constructed. We construct M_{k+1} as follows. Let $V = \{v_1, \dots, v_n\}$ be the vertex set of M_k. Let $U = \{u_1, \dots, u_n\}$ (where the u_i's are "new" vertices; we think of u_i as a "duplicate" of v_i), and let w be another "new" vertex. Let M_{k+1} have vertex set $V \cup U \cup \{w\}$ and adjacency as follows:

- adjacency between the v_i's is inherited from M_k, that is, $M_{k+1}[V] = M_k$;
- for all $i \in \{1, \dots, n\}$, u_i is non-adjacent to v_i;
- for all distinct $i, j \in \{1, \dots, n\}$, u_i is adjacent to v_j in M_{k+1} if and only if v_i is adjacent to v_j in M_k;
- U is a stable set in M_{k+1};

[1] As usual, $\omega(G)$ is the *clique number* of G, i.e. the maximum size of a clique in G.

- w is adjacent to all vertices in U and non-adjacent to all vertices in V.

The first three Mycielski graphs are represented below.



M_2 M_3 M_4

Lemma 1.1. *For all integers $k \geq 2$, M_k satisfies $\omega(M_k) = 2$ and $\chi(M_k) = k$.*

Proof. We proceed by induction on k. Clearly, $\omega(M_2) = 2$ and $\chi(M_2) = 2$. Next, fix an integer $k \geq 2$, and assume inductively that $\omega(M_k) = 2$ and $\chi(M_k) = k$. We must show that $\omega(M_{k+1}) = 2$ and $\chi(M_{k+1}) = k + 1$. Let $V = \{v_1, \dots, v_n\}$, $U = \{u_1, \dots, u_n\}$, and w be as in the definition of M_{k+1}.

We first show that $\omega(M_{k+1}) = 2$. Since $\omega(M_k) = 2$, and M_k is a subgraph of M_{k+1}, it is clear that $\omega(M_{k+1}) \geq 2$. It remains to show that M_{k+1} is triangle-free. Suppose otherwise, and let T be a triangle in M_{k+1}. Since U is a stable set of G, we see that $|T \cap U| \leq 1$. Since $N_{M_{k+1}}(w) = U$, and since U is a stable set, we further see that $w \notin T$. Finally, since $M_{k+1}[V] = M_k$, and since M_k is triangle-free (by the induction hypothesis), we see that $T \not\subseteq V$. It now follows that $|T \cap U| = 1$ and $|T \cap V| = 2$. Let $p, q, r \in \{1, \dots, k\}$ (with $q \neq r$) be such that $T = \{u_p, v_q, v_r\}$. By the construction of M_{k+1}, $u_p v_p \notin E(M_{k+1})$; since T is a triangle, it follows that $p \notin \{q, r\}$. Since $u_p v_q \in E(M_{k+1})$, it follows from the construction of M_{k+1} that $v_p v_q \in E(M_k)$; similarly, $v_p v_r \in E(M_k)$. But now $\{v_p, v_q, v_r\}$ is a triangle in M_k, a contradiction. So, M_{k+1} is triangle-free, and we deduce that $\omega(M_{k+1}) = 2$.

We now show that $\chi(M_{k+1}) = k + 1$. Let us first show that $\chi(M_{k+1}) \leq k + 1$. First, we properly color M_k with colors $1, \dots, k$ (this is possible because $\chi(M_k) = k$). Next, for each $i \in \{1, \dots, n\}$, we assign to u_i the same color as to v_i. Finally, we assign color $k + 1$ to w. Clearly, this is a proper coloring of M_{k+1}, and it follows that $\chi(M_{k+1}) \leq k + 1$.

Finally, we show that $\chi(M_{k+1}) \geq k + 1$. Suppose otherwise, that is, suppose that $\chi(M_{k+1}) \leq k$. Fix a proper coloring $c : V(M_{k+1}) \to \{1, \dots, k\}$ of M_{k+1}. We will use the coloring c of M_{k+1} to construct a proper $(k-1)$-coloring of M_k, which will contradict the fact that $\chi(M_k) = k$. By symmetry, we may assume that $c(w) = k$. Since w is adjacent to every vertex in U, it follows that c does not assign color k to any vertex in U. Now, let V_k be the set of all vertices in V to which c assigns color k. Since c is a proper coloring of M_{k+1}, we know that V_k is a stable set in M_{k+1} (and therefore, in M_k as well). Now, define $c' : V \to \{1, \dots, k-1\}$ as follows:

- $c' \restriction (V \setminus V_k) = c \restriction (V \setminus V_k)$;[2]
- for all $v_i \in V_{k+1}$, set $c'(v_i) = c(u_i)$.

Let us check that c' is a proper coloring of M_k. Fix distinct $i, j \in \{1, \dots, n\}$, and suppose that v_i is adjacent to v_j in M_k. We must show that $c'(v_i) \neq c'(v_j)$. Since V_k is a stable set, we know that at most one of v_i, v_j belongs to V_k. If $v_i, v_j \in V \setminus V_k$, then it follows from the construction of c', and from the fact that c is a proper coloring of M_{k+1} that $c'(v_i) = c(v_i) \neq c(v_j) = c'(v_j)$. It remains to consider the case when exactly one of v_i, v_j belongs to V_k; by symmetry, we may assume that $v_i \in V_k$ and $v_j \in V \setminus V_k$. By the construction of M_{k+1}, u_i is adjacent to v_j in M_{k+1}, and so $c(u_i) \neq c(v_j)$. But now by the construction of c', we have that $c'(v_i) = c(u_i) \neq c(v_j) = c'(v_j)$, which is what we needed. Thus, c' is a proper $(k-1)$-coloring of M_k, contrary to the fact that $\chi(M_k) = k$. □

As an immediate corollary of Lemma 1.1, we get the following.

Theorem 1.2. *There exist triangle-free graphs of arbitrarily large chromatic number. More precisely, for every positive integer k, there exists a graph G such that $\omega(G) = 2$ and $\chi(G) \geq k$.*

Proof. This follows from Lemma 1.1. □

We remark that Erdős (1961) applied the probabilistic method to demonstrate the existence of graphs with arbitrarily high girth and chromatic number (the *girth* of a graph G that has at least one cycle is the length of the shortest cycle in G). Graphs of high girth are triangle-free, and so this result of Erdős is stronger than Theorem 1.2.

2 Perfect graphs

In the previous section, we saw that there exist graphs of small clique number, but large chromatic number. At the other extreme, we might consider graphs for which $\chi = \omega$. This, however, turns out not to be a very interesting question. Indeed, suppose H is any graph at all, and let G be the disjoint union of H and $K_{\chi(H)}$; then $\chi(G) = \omega(G)$, but we can say very little about the structure of G (since G was built starting from an arbitrary graph H).

Here is a more interesting definition. A graph is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$.[3]

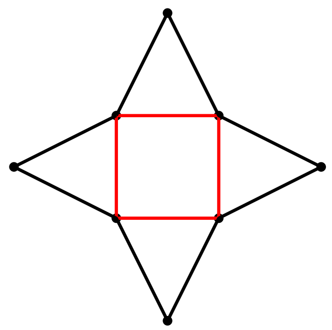
Since every graph is an induced subgraph of itself, we see that every perfect graph G satisfies $\chi(G) = \omega(G)$. Importantly, though, in a perfect graph, $\chi = \omega$ should hold not only for the graph itself, but also for all its induced subgraphs.

[2] This means that $c'(v_i) = c(v_i)$ for all $v_i \in V \setminus V_k$.

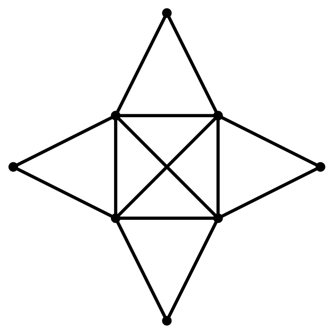
[3] A graph H is an *induced subgraph* of a graph G if $V(H) \subseteq V(G)$, and for all distinct $u, v \in V(H)$, we have that $uv \in E(H)$ if and only if $uv \in E(G)$.

3 Chordal graphs

In this section, we consider a particular subclass of perfect graphs, called "chordal" graphs. A graph is *chordal* (or *triangulated*) if every cycle of length strictly greater than three has a chord (a *chord* of a cycle is an edge joining two nonconsecutive vertices of the cycle). In other words, a graph is *chordal* if it contains no induced cycles of length at least four. For example, in the picture below, the graph on the left is not chordal (because it contains an induced cycle of length four, in red), whereas the one on the right is chordal (this graph contains a cycle of length four, but the cycle is not induced).



not chordal



chordal

Note that all induced subgraphs of a chordal graph are chordal.

Chordal graphs were one of the first classes of graphs to be recognized as perfect; the study of chordal graphs can be seen as the beginning of the theory of perfect graphs. As we shall see, there are efficient algorithms for recognizing chordal graphs and for solving the vertex coloring and related optimization problems on chordal graphs. In many applications of vertex-coloring, the graphs actually are chordal.

In this section, a *cutset* of a graph is a set of vertices whose deletion yields a disconnected graph. More precisely, a *cutset* of a graph G is a (possibly empty) set $S \subsetneqq V(G)$ such that $G \setminus S$ is disconnected.[4] A *clique-cutset* is a cutset that is a clique, that is, a *clique-cutset* of a graph G is a clique $C \subsetneqq V(G)$ of G such that $G \setminus C$ is disconnected.[5]

3.1 Characterizing chordal graphs

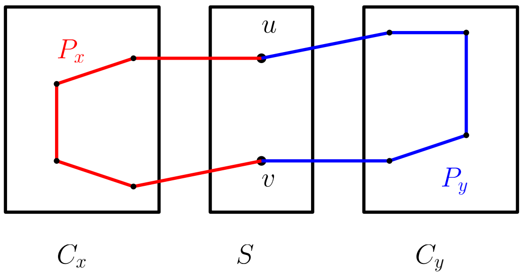
Lemma 3.1. *Let G be a chordal graph that is not complete, let x and y be non-adjacent vertices of G, and let S be a minimal cutset of G separating x and y.*[6] *Then S is a clique of G.*

[4] Sometimes, a cutset is defined to be a set of vertices whose deletion increases the number of components, but that definition is inconvenient in this context.

[5] In particular, if G is disconnected, then $\emptyset$ is a clique-cutset of G.

[6] This means that $S \subseteq V(G) \setminus \{x, y\}$, x and y are in distinct components of $G \setminus S$, and for all $S' \subsetneqq S$, x and y are in the same component of $G \setminus S'$.

Proof. Suppose that S is not a clique, and let u and v be two nonadjacent vertices of S. Let C_x be the component of $G \setminus S$ that contains x, and let C_y be the component of $G \setminus S$ that contains y. By the minimality of S, every vertex of S has a neighbor both in C_x and in C_y. Now, suppose that S is not a clique, and fix distinct, non-adjacent vertices $u, v \in S$. Let P_x be a minimum-length path between u and v in $G[V(C_x) \cup \{u, v\}]$, and let P_y be a minimum-length path between u and v in $G[V(C_y) \cup \{u, v\}]$.[7]



By the minimality of P_x and P_y, we see that both P_x and P_y are induced paths of G, and since u and v are non-adjacent, we see that each of them has at least two edges. Since the interior of P_x belongs to C_x, and the interior of P_y belongs to C_y, we see that there are no edges between the interiors of P_x and P_y. Thus, $P_x \cup P_y$ is an induced cycle of length at least four in G, a contradiction. □

Theorem 3.2. *If G is a chordal graph, then either G is a complete graph or G admits a clique-cutset.*

Proof. Let G be a chordal graph that is not complete. Let x and y be non-adjacent vertices of G, and let S be a minimal cutset of G separating x from y.[8] By Lemma 3.1, S is a clique. It follows that S is a clique-cutset of G. □

Corollary 3.3. *Chordal graphs are perfect.*

Proof. Since every induced subgraph of a chordal graph is chordal, it is enough to show that every chordal graph G satisfies $\chi(G) = \omega(G)$.[9] So, fix a chordal graph G, and assume inductively that all chordal graphs G' on fewer than $|V(G)|$ vertices satisfy $\chi(G') = \omega(G')$. We must show that $\chi(G) = \omega(G)$. If G is a complete graph, then it is clear that $\chi(G) = \omega(G)$.

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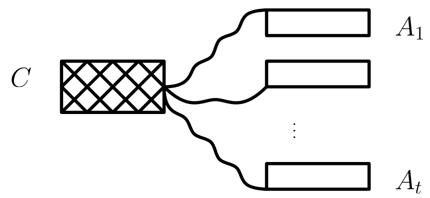
[7] $G[V(C_x) \cup \{u, v\}]$ is connected because C_x is connected, and both u and v have a neighbor in C_x. Similarly, $G[V(C_y) \cup \{u, v\}]$ is connected. So, P_x and P_y exist.

[8] To see that S exists, we first observe that $V(G) \setminus \{x, y\}$ is a cutset of G separating x from y. Of all subsets of $V(G) \setminus \{x, y\}$ separating x from y, let S be one that has a minimum number of vertices.

[9] Indeed, suppose we have shown that all chordal graphs G satisfy $\chi(G) = \omega(G)$. Now, fix a chordal graph G, and let H be an induced subgraph of G. Then H is chordal, and so $\chi(H) = \omega(H)$. So, G is perfect.

So, assume that G is not complete. Then by Theorem 3.2, G admits a clique-cutset, call it C. Let $A_1, \dots, A_t$ ($t \geq 2$) be the vertex sets of the components of $G \setminus C$.



For all $i \in \{1, \dots, t\}$, let $G_i := G[A_i \cup C]$. Note that every clique of G is in fact a clique of one of $G_1, \dots, G_t$,[10] and it follows that $\omega(G) = \max\{\omega(G_1), \dots, \omega(G_t)\}$. On the other hand, by Lemma 2.1 from Lecture Notes 4, we have that $\chi(G) = \max\{\chi(G_1), \dots, \chi(G_t)\}$. Finally, for all $i \in \{1, \dots, t\}$, the induction hypothesis guarantees that $\chi(G_i) = \omega(G_i)$. So,

$$\begin{aligned} \chi(G) &= \max\{\chi(G_1), \dots, \chi(G_t)\} \\ &= \max\{\omega(G_1), \dots, \omega(G_t)\} \\ &= \omega(G), \end{aligned}$$

which is what we needed. □

3.2 Simplicial vertices

A vertex x of a graph G is *simplicial* if $N_G(x)$ is a clique of G.

Theorem 3.4 (Dirac, 1961)**.** *Every chordal graph has a simplicial vertex. Moreover, every chordal graph that is not complete has (at least) two non-adjacent simplicial vertices.*

Proof. We proceed by induction on the number of vertices. Let G be a chordal graph, and assume inductively that the claim holds for chordal graphs on fewer than $|V(G)|$ vertices.[11] We must show that the claim holds for G. If G is a complete graph, then clearly, any vertex of G is simplicial. So assume that G is not complete (and in particular, $|V(G)| \geq 2$). By Theorem 3.2, G contains a clique-cutset, call it C. Let A and B be the vertex-sets of two distinct components of G, and set $G_A = G[A \cup C]$ and $G_B = G[B \cup C]$.

[10] This is because there are no edges between any two of the sets $A_1, \dots, A_t$, and so no clique of G intersects more than one of $A_1, \dots, A_t$.

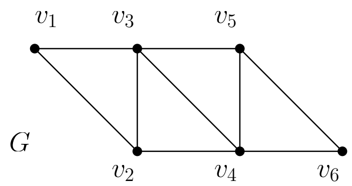
[11] More precisely, we assume inductively that for every chordal graph G' such that $|V(G')| < |V(G)|$, G' has a simplicial vertex, and furthermore, if G' is not a complete graph, then G' has two non-adjacent simplicial vertices.

> **Claim.** A contains a vertex that is simplicial in G_A, and B contains a vertex that is simplicial in G_B.

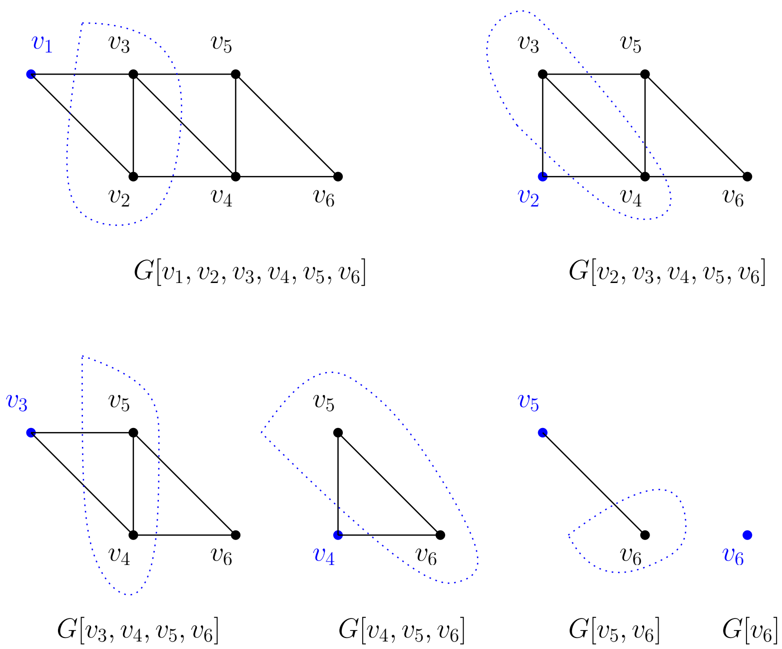
Proof of the Claim. By symmetry, it suffices to show this for A. If G_A is complete, then any vertex in A is simplicial in G_A. Otherwise, by the induction hypothesis, G_A contains two non-adjacent simplicial vertices; since C is a clique, C may contain at most one of these two vertices, and consequently, A contains the other (possibly, A contains both of them). This proves the Claim. ♦

Now, using the Claim, we let $a \in A$ be a simplicial vertex of G_A, and we let $b \in B$ be a simplicial vertex of G_B. Clearly, a and b are non-adjacent. Furthermore, we have that $N_G(a) = N_{G_A}(a)$ and $N_G(b) = N_{G_B}(b)$, and we deduce that a and b are simplicial vertices of G. □

A *simplicial elimination ordering* (sometimes also called a *perfect elimination ordering*) of a graph G is an ordering $v_1, \dots, v_n$ of its vertices such that for all $i \in \{1, \dots, n\}$, v_i is simplicial in the graph $G[v_i, \dots, v_n]$. For instance, $v_1, \dots, v_6$ is a simplicial elimination ordering of the graph G in the picture below.



Indeed, consider the picture below. Clearly, for each $i \in \{1, \dots, 6\}$, v_i is simplicial in $G[v_i, \dots, v_6]$.



Theorem 3.5 (Fulkerson and Gross, 1965)**.** *For a graph G, the following statements are equivalent:*

(i) G is chordal;

(ii) G has a simplicial elimination ordering;

(iii) for all non-adjacent vertices x and y of G, every minimal cutset of G separating x from y is a clique.

Proof. **(i) $\Rightarrow$ (iii):** This follows from Lemma 3.1.

(iii) $\Rightarrow$ (i): We prove the contrapositive: if (i) if false, then (iii) is false. So assume that (i) is false, that is, that G is not chordal. Let C be an induced cycle of length at least four in G, let x and y be non-adjacent vertices of C, and let P_1 and P_2 be the two paths between x and y in C; clearly, each of P_1, P_2 has at least two edges, and in particular, $V(P_1) \setminus \{x, y\}$ and $V(P_2) \setminus \{x, y\}$ are non-empty. Let S be a minimal cutset of G separating x from y. Clearly, S must intersect both $V(P_1) \setminus \{x, y\}$ and $V(P_2) \setminus \{x, y\}$. But since C is a chordless cycle, we know that there are no edges between $V(P_1) \setminus \{x, y\}$ and $V(P_2) \setminus \{x, y\}$, and it follows that S is not a clique. Thus, (iii) is false.

(i) $\Rightarrow$ (ii): We proceed by induction on the number of vertices. Clearly, the claim holds for one-vertex graphs. Now, fix a positive integer n, and assume that the claim holds for all chordal graphs on n vertices. Let H be a chordal graph on $n + 1$ vertices. By Theorem 3.4, H has at least one simplicial vertex, call it x_0. Then $H \setminus x_0$ is a chordal graph on n vertices, and so by the induction hypothesis, $H \setminus x_0$ has a simplicial elimination ordering, say $x_1, \dots, x_n$. But now $x_0, x_1, \dots, x_n$ is a simplicial elimination ordering of H.

(ii) $\Rightarrow$ (i): Suppose that $v_1, \dots, v_n$ is a simplicial elimination ordering of G; we claim that G is chordal. Let C be an induced cycle of G; we must show that C is a triangle. Let $x = v_i$ be the lowest-indexed vertex from our simplicial elimination ordering that belongs to the cycle C, and let y, z be the two neighbors of x in C. Since $x = v_i$ is simplicial in $G[v_i, v_{i+1}, \dots, v_n]$, since y, z are distinct neighbors of x, and since (by the minimality of i) we have that $y, z \in \{v_{i+1}, \dots, v_n\}$, we see that $yz \in E(G)$. Since C is an induced cycle, it follows that C is a triangle. This proves that G is chordal. □

Note that Theorem 3.5 gives an $O(n^4)$ time recognition algorithm for chordal graphs (we repeatedly search for simplicial vertices). In fact, chordal graphs can be recognized in $O(n + m)$ time using the so called Lexicographic breadth-first-search (LexBFS) due to Rose, Tarjan, and Lueker (1976), but we omit the details.

3.3 Efficient algorithms for chordal graphs

In this subsection, G is a chordal graph on n vertices, and $v_1, \dots, v_n$ is a simplicial elimination ordering on G.[12] For each $i \in \{1, \dots, n\}$, set $X_i := N_G[v_i] \cap \{v_i, \dots, v_n\}$;[13] so, X_i is the closed neighborhood of v_i in the graph $G[v_i, \dots, v_n]$.

Lemma 3.6. *$X_1, \dots, X_n$ are all cliques of G. Furthermore, for every maximal clique C of G, there exists some $i \in \{1, \dots, n\}$ such that $C = X_i$.*[14]

Proof. The fact that the sets X_i are cliques follows immediately from the definition of a simplicial elimination ordering and the construction of the sets X_i. Now, let C be a maximal clique of G. Let $i \in \{1, \dots, n\}$ be minimal with $v_i \in C$. Then clearly, $C \subseteq X_i$. Since C is a maximal clique, and X_i is a clique, it follows that $C = X_i$. □

Lemma 3.7 (Fulkerson and Gross, 1965)**.** *G has at most n maximal cliques. Furthermore, equality holds if and only if G is edgeless.*

Proof. The fact that G has at most n maximal cliques follows immediately from Lemma 3.6. Clearly, if G is edgeless, then G has precisely n maximal cliques (indeed, each one-vertex subset of $V(G)$ is a maximal clique of G). Suppose now that G has at least one edge; let $i \in \{1, \dots, n\}$ be the largest index such that v_i has a neighbor in G. Let v_j be a neighbor of v_i in G; by the maximality of i, we have that $j < i$. Then $X_i = \{v_i\}$ and $\{v_j, v_i\} \subseteq X_j$, and so $X_i \subsetneqq X_j$. By Lemma 3.6, both X_i and X_j are cliques. So, X_i is *not* a maximal clique of G, and Lemma 3.6 implies that G has fewer than n maximal cliques. □

A *clique cover* of a graph H is a partition of $V(H)$ into cliques. The *clique cover number* of H, denoted by $\overline{\chi}(H)$, is the smallest size of a clique cover of H; a *minimum clique cover* of H is a clique cover of size precisely $\overline{\chi}(H)$. Note that every graph H satisfies $\alpha(H) \leq \overline{\chi}(H)$. Moreover, since proper colorings correspond to partitions of the vertex set into stable sets (color classes), it is clear that every graph H satisfies $\overline{\chi}(H) = \chi(\overline{H})$.

We define a (finite) sequence $i_1, \dots, i_t$ as follows. First, let $i_1 := 1$. Once $i_1, \dots, i_{j-1}$ have been defined, we either terminate or extend the sequence, as follows. If $V(G) = X_{i_1} \cup \dots \cup X_{i_{j-1}}$, then we set $t = j - 1$, and we terminate the sequence; otherwise, we let $i_j \in \{1, \dots, n\}$ be the smallest index such that $v_{i_j} \notin X_{i_1} \cup \dots \cup X_{i_{j-1}}$. Set $Y_1 := X_{i_1}$, and for all $j \in \{2, \dots, t\}$, set $Y_j := X_{i_j} \setminus (Y_1 \cup \dots \cup Y_{j-1})$.

[12] By Theorem 3.5, every chordal graph has a simplicial elimination ordering, and clearly, we can find such an ordering in polynomial time.

[13] As usual, for a graph G and a vertex $x \in V(G)$, we denote by $N_G(x)$ the set of all neighbors of x in G, and we set $N_G[x] = \{x\} \cup N_G(x)$. So, $N_G(x)$ is the open neighborhood (or simply neighborhood) of x in G, and $N_G[x]$ is the closed neighborhood of x in G.

[14] However, not all X_i's need be maximal cliques.

Theorem 3.8 (Gavril, 1972). *The set $\{v_{i_1}, \dots, v_{i_t}\}$ is a maximum stable set of G, and $(Y_1, \dots, Y_t)$ is a minimum clique cover of G.*

Proof. First of all, note that $i_1 < \cdots < i_t$. Suppose that $v_p v_q \in E(G)$ for some $p, q \in \{i_1, \dots, i_j\}$, with $p < q$; then $v_q \in X_p$, contrary to the choice of q. Thus, $\{v_{i_1}, \dots, v_{i_t}\}$ is a stable set of size t, and we deduce that $t \leq \alpha(G)$.

Further, it is clear that $Y_1, \dots, Y_t$ are pairwise disjoint cliques.[15] It is also clear that $V(G) = Y_1 \cup \cdots \cup Y_t$, for otherwise, we could extend the sequence $i_1, \dots, i_t$.[16] Thus, $(Y_1, \dots, Y_t)$ is a clique cover of G, and it follows that $\overline{\chi}(G) \leq t$.

We now have that $t \leq \alpha(G) \leq \overline{\chi}(G) \leq t$, and it follows that $\alpha(G) = \overline{\chi}(G) = t$. Thus, $\{y_1, \dots, y_t\}$ is a maximum stable set of G, and $(Y_1, \dots, Y_t)$ is a minimum clique cover of G. □

Lemma 3.9. *G can be optimally colored (i.e. properly colored using precisely $\chi(G)$ colors) by applying the greedy coloring algorithm to G with the ordering $v_n, \dots, v_1$.*[17]

Proof. Clearly, the greedy coloring produces a proper coloring of G. If we apply the greedy coloring algorithm to G with the ordering $v_n, \dots, v_1$, then when we reach a vertex v_i, the neighbors of v_i that have already been colored are precisely those from the clique $X_i \setminus \{v_i\}$, and consequently, at most $\omega(G) - 1$ neighbors of v_i have already been colored.[18] Thus, the greedy algorithm applied to G with this ordering uses no more than $\omega(G)$ colors. Since every graph H satisfies $\chi(H) \geq \omega(H)$, it follows that the greedy coloring algorithm used precisely $\omega(G)$ colors, and that the coloring that it produced is optimal. □

Clearly, Lemma 3.6, Theorem 3.8, and Lemma 3.9 yield polynomial time algorithms for finding a maximum clique, a maximum stable set, a minimum clique-cover, and an optimal coloring of a chordal graph.

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[15] Indeed, for all $j \in \{1, \dots, t\}$, we have that $Y_j \subseteq X_{i_j}$, and by Lemma 3.6, X_{i_j} is a clique. The fact that $Y_1, \dots, Y_t$ are pairwise disjoint follows from the construction of $Y_1, \dots, Y_t$.

[16] We are using the fact that $Y_1 \cup \cdots \cup Y_t = X_{i_1} \cup \cdots \cup X_{i_t}$.

[17] So, we are using the reverse of our simplicial elimination ordering.

[18] Indeed, X_i is a clique, and the size of this clique is at most $\omega(G)$. So, $|X_i \setminus \{v_i\}| \leq \omega(G) - 1$.