NDMI012: Combinatorics and Graph Theory 2

Lecture #6

Vertex and edge coloring: Brooks' theorem and Vizing's theorem

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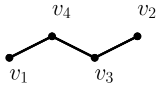
March 22, 2022

- A *greedy* coloring of a graph G with vertex ordering $V(G) = \{v_1, \dots, v_n\}$ is a coloring of G obtained as follows: for each $i \in \{1, \dots, n\}$, we assign to v_i the smallest positive integer that was not used on any smaller-indexed neighbor of v_i.
- For example, the greedy coloring applied to the graph below, with the ordering v_1, v_2, v_3, v_4, yields the coloring $c(v_1) = 1$, $c(v_2) = 1$, $c(v_3) = 2$, and $c(v_4) = 3$.



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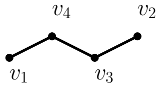

- The greedy coloring of a graph G always produces a proper coloring of G, but the coloring need not be optimal, i.e. it may use more than $\chi(G)$ colors.

Lemma 1.1

Every graph G satisfies $\chi(G) \leq \Delta(G) + 1$.

Proof. A greedy coloring of a graph G (using any ordering of $V(G)$) produces a proper coloring of G that uses at most $\Delta(G) + 1$ colors; so, $\chi(G) \leq \Delta(G) + 1$.

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Brooks' theorem

Let G be a connected graph that is neither a complete graph nor an odd cycle. Then $\chi(G) \leq \Delta(G)$.

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- First, we prove a technical lemma.

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If G is connected and not regular, then $\chi(G) \leq \Delta(G)$.

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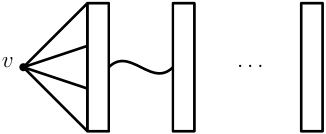
Proof (outline). Let G be a connected graph that is not regular, and fix a vertex $v \in V(G)$ such that $d_G(v) \leq \Delta(G) - 1$. We order $V(G)$ according to the distance from v, that is, we list v first, then we list all vertices at distance one from v (in any order), then we list all vertices at distance two from v (in any order), etc. Let $v_1, \dots, v_n$ be the resulting ordering of G.



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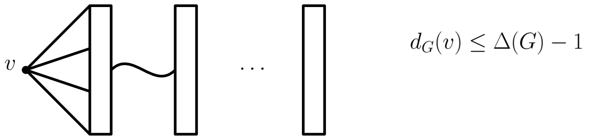

We now color G greedily using the ordering $v_n, \dots, v_1$, and we obtain a proper coloring of G that uses at most $\Delta(G)$ colors.

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Proof (outline). To simplify notation, we set $\Delta := \Delta(G)$. WTS $\chi(G) \leq \Delta$.

Since G is connected and not complete, we see that $\Delta \geq 2$. Next, suppose that $\Delta = 2$. Since G is connected, it follows that G is either a path on at least two edges or a cycle. But by hypothesis, G is not an odd cycle, and so G is either a path on at least two edges or an even cycle. It is now obvious that $\chi(G) \leq \Delta$.

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From now on, we assume that $\Delta \geq 3$. Note that this implies that $|V(G)| \geq 4$. We may further assume that G is regular, for otherwise, we are done by Lemma 1.2.

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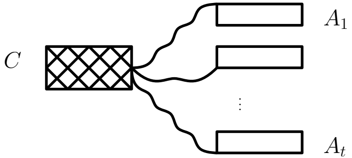
Claim 1. *If G has a clique-cutset, then $\chi(G) \leq \Delta$.*

Proof of Claim 1 (outline). Suppose that G has a clique-cutset, and let C be a minimal clique-cutset of G. Let $A_1, \dots, A_t$ $(t \geq 2)$ be the vertex sets of the components of $G \setminus C$. For all $i \in \{1, \dots, t\}$, let $G_i := G[A_i \cup C]$.



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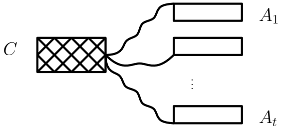
Proof of Claim 1 (outline, continued).



$$\chi(G) = \max\{\chi(G_1), \ldots, \chi(G_t)\} \quad \text{by Lemma 2.1 from Lecture Notes 4}$$

$$\leq \Delta(G) \quad \text{by Lemma 1.2}$$

This proves Claim 1.

Brooks' theorem

Let G be a connected graph that is neither a complete graph nor an odd cycle. Then $\chi(G) \leq \Delta(G)$.

Proof (outline). Reminder: G is Δ-regular, $\Delta \geq 3$.

Claim 2. *If G is not 3-connected, then $\chi(G) \leq \Delta$.*

Proof of Claim 2 (outline).

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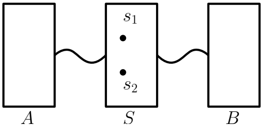
Proof of Claim 2 (outline). Assume that G is not 3-connected.



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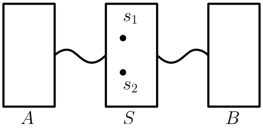

s_1 has a neighbor in both A and B, and it has at least two neighbors in at least one of them. (Same for s_2.)

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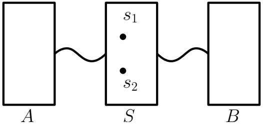

s_1 has a neighbor in both A and B, and it has at least two neighbors in at least one of them. (Same for s_2.) WMA either

- s_1 has at least two neighbors in A, and s_2 has at least two neighbors in B, or
- s_1, s_2 each have exactly one neighbor in A, and at least two neighbors in B.

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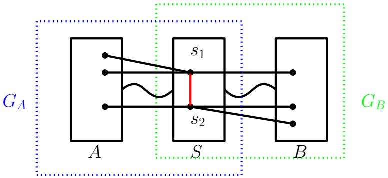
Proof of Claim 2 (outline, continued). Suppose s_1 has at least two neighbors in A, and s_2 has at least two neighbors in B.



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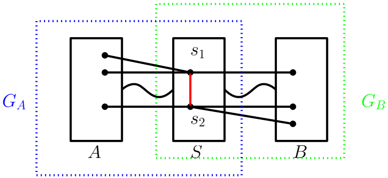

Then $\chi(G) \leq \chi(G + s_1 s_2) = \max\{\chi(G_A), \chi(G_B)\} \leq \Delta(G)$.

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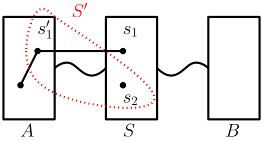
Proof of Claim 2 (outline, continued). The other case (i.e. when s_1, s_2 each have exactly one neighbor in A, and at least two neighbors in B) can be reduced to the previous case. (Details: Lecture Notes.)



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In view of Claim 2, we may now assume that G is 3-connected.

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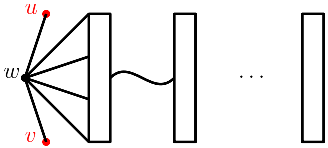
In view of Claim 2, we may now assume that G is 3-connected. Since G is connected and not complete, G has two vertices, call them u and v, at distance two from each other; let w be a common neighbor of u and v.



Since G is 3-connected, $G' := G \setminus \{u, v\}$ is connected. We now order $V(G')$ according to the distance from w (starting with w), and we add u, v at the end of our list. This produces an ordering $v_1, \dots, v_n$ of $V(G)$ (with $v_1 = w$, $v_{n-1} = u$, and $v_n = v$).

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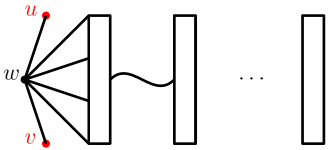
Proof (outline). Reminder: G is Δ-regular, $\Delta \geq 3$, G is 3-connected.



We now color G greedily using the ordering $v_n, \ldots, v_1$. This produces a proper coloring of G that uses at most Δ colors.

Definition

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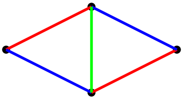
Theorem 2.1

A connected graph is Eulerian if and only if all its vertices are of even degree.

Proof. Discrete Math.

Definition

A *k-edge-coloring* of a graph G is a mapping $c : E(G) \to C$, with $|C| = k$. Elements of C are called *colors*. An edge-coloring is *proper* if for any two distinct edges e and f that share an endpoint, we have that $c(e) \neq c(f)$.



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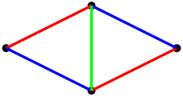

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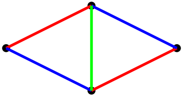

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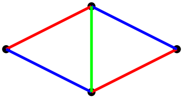
The *edge chromatic number* (or *chromatic index*) of a graph G, denoted by $\chi'(G)$, is the minimum k such that G is k-edge-colorable.



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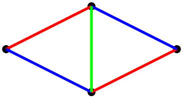
  - Consequently, $\chi'(G) \geq \Delta(G)$.



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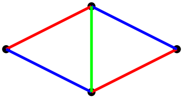
  - Sets $E_1, \ldots, E_k$ are called *color classes*.
  - A proper k-edge-coloring is one where each E_i is a matching.



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Every graph G satisfied $\chi'(G)\nu(G) \geq |E(G)|$. Consequently, if G has at least one edge, then $\chi'(G) \geq \left\lceil \frac{|E(G)|}{\nu(G)} \right\rceil$.

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$$\begin{aligned} |E(G)| &= \textstyle\sum_{i=1}^{k} |E_i| && \text{because } (E_1, \ldots, E_k) \text{ is a partition of } E(G) \\ &\leq \textstyle\sum_{i=1}^{k} \nu(G) && \text{because } E_1, \ldots, E_k \text{ are matchings of } G \\ &= k\nu(G) \\ &= \chi'(G)\nu(G). \end{aligned}$$

This proves that $\chi'(G)\nu(G) \geq |E(G)|$.

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Proof (continued). Reminder: $\chi'(G)\nu(G) \geq |E(G)|$

If G has at least one edge, then clearly, $\nu(G) \geq 1$, and we deduce that $\chi'(G) \geq \frac{|E(G)|}{\nu(G)}$; since $\chi'(G)$ is an integer, it follows that $\chi'(G) \geq \left\lceil \frac{|E(G)|}{\nu(G)} \right\rceil$.

- Our goal is to prove the following two theorems.

Theorem 3.4

If G is a bipartite graph, then $\chi'(G) = \Delta(G)$.

Vizing's theorem

Every graph G satisfies $\chi'(G) \leq \Delta(G) + 1$.

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If G is a bipartite graph, then $\chi'(G) = \Delta(G)$.

Vizing's theorem

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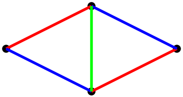
- First, we need some definitions and technical lemmas.

- Given a (not necessarily proper) edge-coloring of a graph G, we say that color i is *represented* at a vertex v of G if some edge incident with v has color i.



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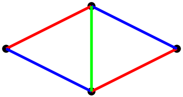

Lemma 3.2

Let G be a connected graph that is not an odd cycle. Then G has a (not necessarily proper) 2-edge-coloring in which both colors are represented at each vertex of degree at least 2.

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Proof (outline).

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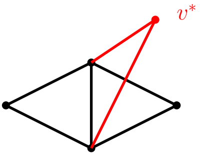

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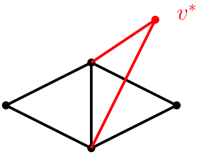

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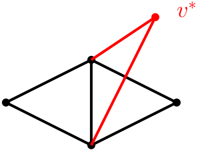

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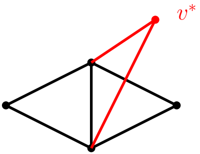

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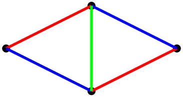
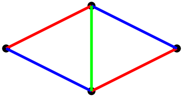


- Given a (not necessarily proper) k-edge-coloring $\mathcal{C}$ and a vertex v of G, we denote by $c_{\mathcal{C}}(v)$ the number of distinct colors represented at v.



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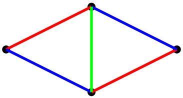

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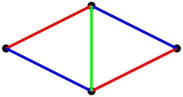

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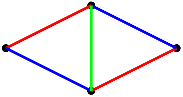
$$\sum_{v \in V(G)} c_{\mathcal{C}'}(v) > \sum_{v \in V(G)} c_{\mathcal{C}}(v).$$



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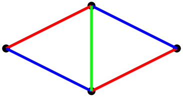

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- Note that any proper edge-coloring of a graph G is unimprovable. However, the converse does not hold in general.

Lemma 3.3

Let $\mathcal{C} = (E_1, \ldots, E_k)$ be an unimprovable k-edge-coloring of a graph G. If there is a vertex u of G and colors i and j such that i is not represented at u and j is represented at least twice at u, then the component of $G[E_i \cup E_j]$ that contains u is an odd cycle.

Proof (outline).

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Recolor the edges of H with colors i and j in this way to get a new k-edge-coloring $\mathcal{C}' = (E'_1, \ldots, E'_k)$ of G. Then the resulting coloring is an improvement of $\mathcal{C}$, a contradiction.

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Let vertex $u \in V(G)$ and colors $i_0, i_1 \in \{1, \dots, \Delta + 1\}$ be such that i_0 is not represented at u, and i_1 is represented at least twice at u.

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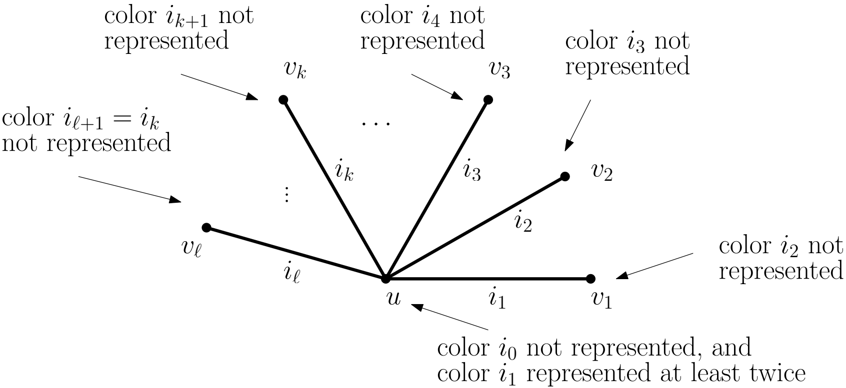

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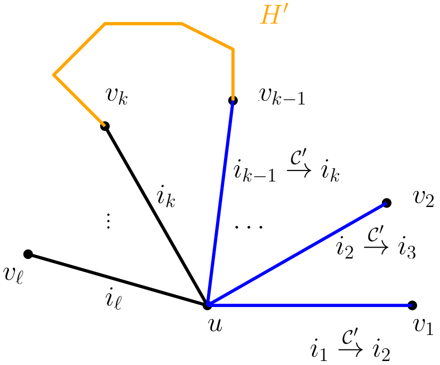

Let H' be the component of $G[E'_{i_0} \cup E'_{i_k}]$ that contains u. By Lemma 3.3, H' is an odd cycle.

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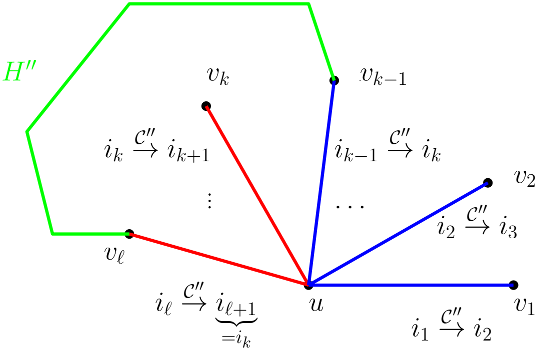
Proof (outline, continued).



Let H'' be the component of $G[E'_{i_0} \cup E'_{i_k}]$ that contains u. By Lemma 3.3, H'' is an odd cycle.

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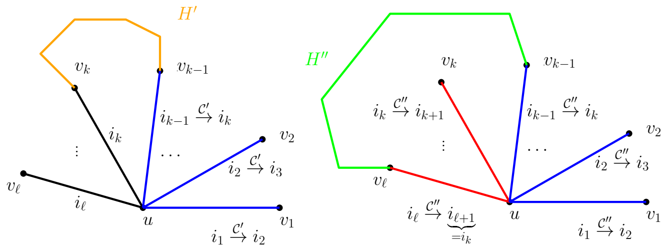

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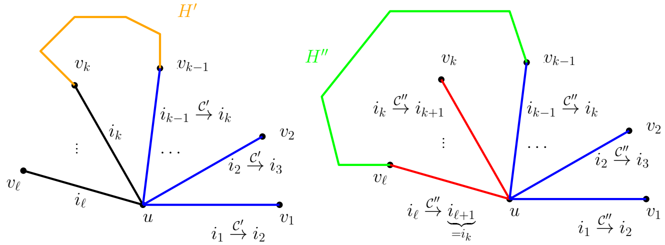

But H' and H'' are the same, except for one edge! This is impossible because they are both odd cycles.

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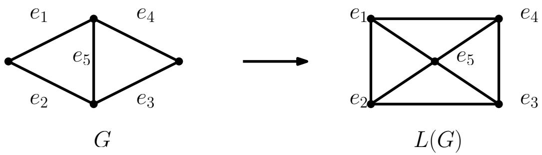
- It is NP-complete to decide whether $\chi' = \Delta$ (even when $\Delta = 3$). We omit the details.

Definition

Given a graph G, the *line graph* of G, denoted by $L(G)$, is the graph with vertex set $E(G)$, in which distinct $e, f \in E(G)$ are adjacent if and only if they share an endpoint in G.



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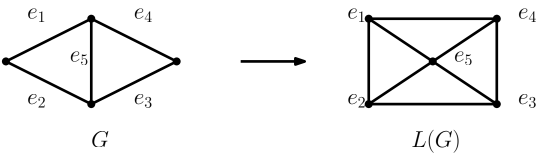

- Obviously, $\chi(L(G)) = \chi'(G)$.

Lemma 3.6

Every graph G satisfies $\chi(L(G)) \leq \omega(L(G)) + 1$.

Proof.

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Proof. Let G be a graph. Then clearly, $\chi(L(G)) = \chi'(G)$. Furthermore, for any vertex v, the set of all edges incident with v in G is a clique of size $d_G(v)$ in $L(G)$; consequently, $\omega(L(G)) \geq \Delta(G)$. But now

$$\begin{aligned} \chi(L(G)) &= \chi'(G) \\ &\leq \Delta(G) + 1 && \text{by Vizing's theorem} \\ &\leq \omega(L(G)) + 1. \end{aligned}$$