NDMI012: Combinatorics and Graph Theory 2

Lecture #5

Graphs on surfaces

Irena Penev

March 15, 2022

Definition

A *surface* is a connected 2-dimensional compact manifold with no boundary.

Definition

A *surface* is a connected 2-dimensional compact manifold with no boundary.

- This definition contains several terms we have not defined, and whose formal definition we omit.

Definition

A *surface* is a connected 2-dimensional compact manifold with no boundary.

- This definition contains several terms we have not defined, and whose formal definition we omit.
- Here is an intuitive explanation:
  - "2-dimensional manifold with no boundary" means that each point has a neighborhood homeomorphic to an open disk;
  - "compact" means that that the surface admits a triangulation with finitely many triangles;
  - "connected" means that there is just one piece.

Definition

A *surface* is a connected 2-dimensional compact manifold with no boundary.

- This definition contains several terms we have not defined, and whose formal definition we omit.
- Here is an intuitive explanation:
  - "2-dimensional manifold with no boundary" means that each point has a neighborhood homeomorphic to an open disk;
  - "compact" means that that the surface admits a triangulation with finitely many triangles;
  - "connected" means that there is just one piece.
- The sphere and the torus are surfaces.

Definition

A *surface* is a connected 2-dimensional compact manifold with no boundary.

- This definition contains several terms we have not defined, and whose formal definition we omit.
- Here is an intuitive explanation:
  - "2-dimensional manifold with no boundary" means that each point has a neighborhood homeomorphic to an open disk;
  - "compact" means that that the surface admits a triangulation with finitely many triangles;
  - "connected" means that there is just one piece.
- The sphere and the torus are surfaces.
- However, the plane is not a surface (because it is not compact).

Definition

A *surface* is a connected 2-dimensional compact manifold with no boundary.

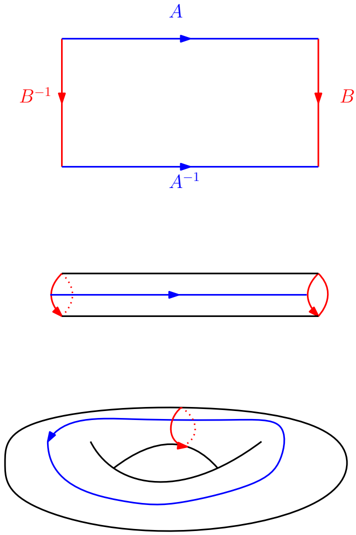
- This definition contains several terms we have not defined, and whose formal definition we omit.
- Here is an intuitive explanation:
  - "2-dimensional manifold with no boundary" means that each point has a neighborhood homeomorphic to an open disk;
  - "compact" means that that the surface admits a triangulation with finitely many triangles;
  - "connected" means that there is just one piece.
- The sphere and the torus are surfaces.
- However, the plane is not a surface (because it is not compact).
- A closed disk is not a surface, either, since it has a boundary.

- In what follows, we consider two surfaces to be the "same" if they are "homeomorphic," that is, if there is a bijection f between them such that both f and f^{-1} are continuous.

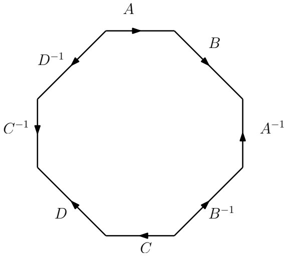
- In what follows, we consider two surfaces to be the “same” if they are “homeomorphic,” that is, if there is a bijection f between them such that both f and f^{-1} are continuous.
- So, if we can obtain one surface from the other by “stretching,” then the two surfaces are the same.

- In what follows, we consider two surfaces to be the “same” if they are “homeomorphic,” that is, if there is a bijection f between them such that both f and f^{-1} are continuous.
- So, if we can obtain one surface from the other by “stretching,” then the two surfaces are the same.
- Thus, a tetrahedron is simply a sphere for our purposes, but a torus is not a sphere.

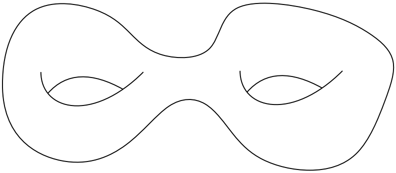
- A torus can be represented by a rectangle (symbolically represented by the string $ABA^{-1}B^{-1}$).



- If we identify corresponding edges in the octagon $ABA^{-1}B^{-1}CDC^{-1}D^{-1}$ below, then we get a double torus (also called the "connected sum of two tori").



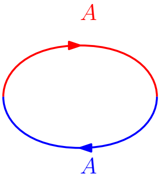
$ABA^{-1}B^{-1}CDC^{-1}D^{-1}$



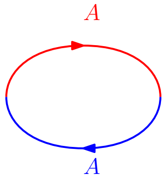
double torus

- The *real projective plane* (or simply *projective plane*) is the surface obtained by starting from the sphere, and then identifying each pair of antipodal points.

- The *real projective plane* (or simply *projective plane*) is the surface obtained by starting from the sphere, and then identifying each pair of antipodal points.
- The projective plane has polygonal representation AA.

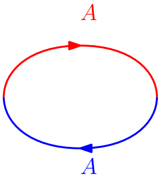


- The *real projective plane* (or simply *projective plane*) is the surface obtained by starting from the sphere, and then identifying each pair of antipodal points.
- The projective plane has polygonal representation AA.



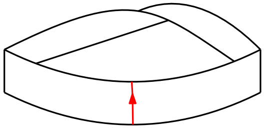
- Unlike the torus, the projective plane cannot be embedded in $\mathbb{R}^3$.

- The *real projective plane* (or simply *projective plane*) is the surface obtained by starting from the sphere, and then identifying each pair of antipodal points.
- The projective plane has polygonal representation AA.

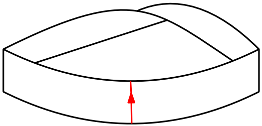


- Unlike the torus, the projective plane cannot be embedded in $\mathbb{R}^3$.
- Still, there is a geometric interpretation (on the next slide).

- Take a rectangle shown below on the left (think of it as a piece of paper), twist it, and identify the two vertical edges (as shown by the arrows). The result (on the bottom right) is called the *Möbius strip*.

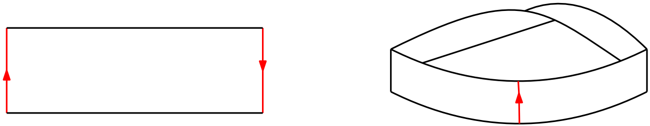


- Take a rectangle shown below on the left (think of it as a piece of paper), twist it, and identify the two vertical edges (as shown by the arrows). The result (on the bottom right) is called the *Möbius strip*.

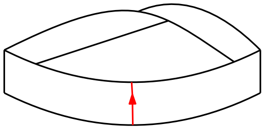


- Note that the boundary of the Möbius strip consists of just one circle, and the Möbius strip has just one "side."

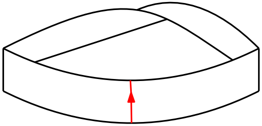
- Take a rectangle shown below on the left (think of it as a piece of paper), twist it, and identify the two vertical edges (as shown by the arrows). The result (on the bottom right) is called the *Möbius strip*.



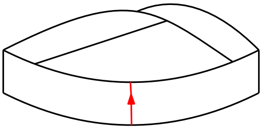
- Note that the boundary of the Möbius strip consists of just one circle, and the Möbius strip has just one "side."
- Now, take a sphere, cut out a small disk from it, and then glue the Möbius strip along the boundary obtained by removing the disk.
- The result is the projective plane (the circle of the Möbius strip corresponds to the edge A from our AA representation of the projective plane).



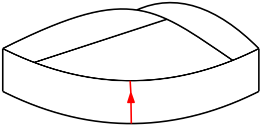
- The projective plane is a type of “non-orientable” surface, which roughly means that we cannot set up a left-right distinction.



- The projective plane is a type of “non-orientable” surface, which roughly means that we cannot set up a left-right distinction.
- Intuitively, imagine a two-dimensional bug on the surface of the Möbius strip (which is part of the projective plane).



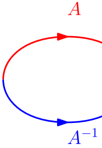
- The projective plane is a type of “non-orientable” surface, which roughly means that we cannot set up a left-right distinction.
- Intuitively, imagine a two-dimensional bug on the surface of the Möbius strip (which is part of the projective plane).
- If the bug keeps going forward, it will eventually come back to the same place (and facing in the same direction), but with left and right reversed.



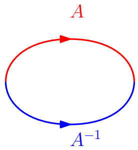
- The projective plane is a type of "non-orientable" surface, which roughly means that we cannot set up a left-right distinction.
- Intuitively, imagine a two-dimensional bug on the surface of the Möbius strip (which is part of the projective plane).
- If the bug keeps going forward, it will eventually come back to the same place (and facing in the same direction), but with left and right reversed.
- This sort of thing is impossible on "orientable" surfaces such as the sphere or the torus (or double torus, triple torus, etc.).

- Now, any $2n$-gon, with edges labeled $A_1, \ldots, A_n$ (in any order), with each letter appearing exactly twice on the $2n$-gon, either in the form A (for clockwise direction) or A^{-1} (for counterclockwise direction) can be transformed into a surface via gluing using the rules described above.

- Now, any $2n$-gon, with edges labeled $A_1, \ldots, A_n$ (in any order), with each letter appearing exactly twice on the $2n$-gon, either in the form A (for clockwise direction) or A^{-1} (for counterclockwise direction) can be transformed into a surface via gluing using the rules described above.
    - Note: AA^{-1} is simply the sphere.

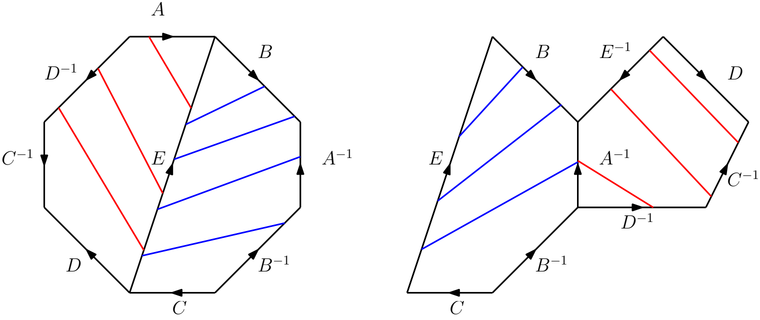


- Now, any $2n$-gon, with edges labeled $A_1, \dots, A_n$ (in any order), with each letter appearing exactly twice on the $2n$-gon, either in the form A (for clockwise direction) or A^{-1} (for counterclockwise direction) can be transformed into a surface via gluing using the rules described above.
  - Note: AA^{-1} is simply the sphere.



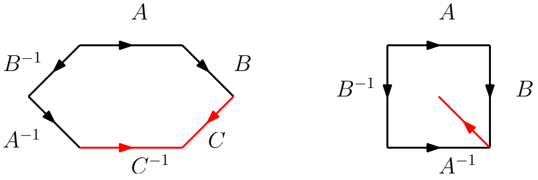
- Some labellings are equivalent (next two slides).

- For example, the two octagons below obviously “encode” the same surface (i.e. after gluing, we get the same surface, in this case, the double torus).



$ABA^{-1}B^{-1}CDC^{-1}D^{-1}$ $BE^{-1}DC^{-1}D^{-1}B^{-1}CE$

- Sometimes, we have “unnecessary” letters/edges in our polygon, as the picture below shows. Both polygons represent a torus.



$ABCC^{-1}A^{-1}B^{-1}$ $ABA^{-1}B^{-1}$

Theorem 1.1

Every surface has a polygonal representation of one of the following forms:

- AA^{-1};
- $(A_1 B_1 A_1^{-1} B_1^{-1})(A_2 B_2 A_2^{-1} B_2^{-1}) \ldots (A_k B_k A_k^{-1} B_k^{-1})$;
- $(A_1 A_1)(A_2 A_2) \ldots (A_k A_k)$.

Proof. Omitted.

Theorem 1.1

Every surface has a polygonal representation of one of the following forms:

- AA^{-1};
- $(A_1 B_1 A_1^{-1} B_1^{-1})(A_2 B_2 A_2^{-1} B_2^{-1}) \dots (A_k B_k A_k^{-1} B_k^{-1})$;
- $(A_1 A_1)(A_2 A_2) \dots (A_k A_k)$.

Proof. Omitted.

- Theorem 1.1 does **not** state that each polygonal representation of a surface has one of the forms from the theorem.

Theorem 1.1

Every surface has a polygonal representation of one of the following forms:

- AA^{-1};
- $(A_1B_1A_1^{-1}B_1^{-1})(A_2B_2A_2^{-1}B_2^{-1})\ldots(A_kB_kA_k^{-1}B_k^{-1})$;
- $(A_1A_1)(A_2A_2)\ldots(A_kA_k)$.

Proof. Omitted.

- Theorem 1.1 does **not** state that each polygonal representation of a surface has one of the forms from the theorem.
- As a matter of fact, each surface has infinitely many polygonal representations (because we can always add AA^{-1}, thus creating a bigger polygon, without changing the surface).

Theorem 1.1

Every surface has a polygonal representation of one of the following forms:

- AA^{-1};
- $(A_1B_1A_1^{-1}B_1^{-1})(A_2B_2A_2^{-1}B_2^{-1})\dots(A_kB_kA_k^{-1}B_k^{-1})$;
- $(A_1A_1)(A_2A_2)\dots(A_kA_k)$.

Proof. Omitted.

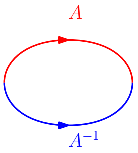
- Theorem 1.1 does **not** state that each polygonal representation of a surface has one of the forms from the theorem.
- As a matter of fact, each surface has infinitely many polygonal representations (because we can always add AA^{-1}, thus creating a bigger polygon, without changing the surface).
- Theorem 1.1 merely states that for each surface S, one of its representations has a "canonical" from (i.e. one of the forms from the theorem).

Theorem 1.1

Every surface has a polygonal representation of one of the following forms:

- AA^{-1};
- $(A_1 B_1 A_1^{-1} B_1^{-1})(A_2 B_2 A_2^{-1} B_2^{-1}) \ldots (A_k B_k A_k^{-1} B_k^{-1})$;
- $(A_1 A_1)(A_2 A_2) \ldots (A_k A_k)$.

- AA^{-1} represents the sphere.



Theorem 1.1

Every surface has a polygonal representation of one of the following forms:

- AA^{-1};
- $(A_1B_1A_1^{-1}B_1^{-1})(A_2B_2A_2^{-1}B_2^{-1})\dots(A_kB_kA_k^{-1}B_k^{-1})$;
- $(A_1A_1)(A_2A_2)\dots(A_kA_k)$.

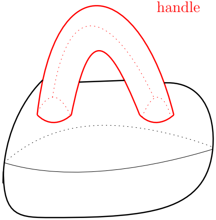
- $(A_1B_1A_1^{-1}B_1^{-1})(A_2B_2A_2^{-1}B_2^{-1})\dots(A_kB_kA_k^{-1}B_k^{-1})$ represents the "connected sum of k tori," i.e. a torus with k holes.

Theorem 1.1

Every surface has a polygonal representation of one of the following forms:

- AA^{-1};
- $(A_1B_1A_1^{-1}B_1^{-1})(A_2B_2A_2^{-1}B_2^{-1})\dots(A_kB_kA_k^{-1}B_k^{-1})$;
- $(A_1A_1)(A_2A_2)\dots(A_kA_k)$.

- $(A_1B_1A_1^{-1}B_1^{-1})(A_2B_2A_2^{-1}B_2^{-1})\dots(A_kB_kA_k^{-1}B_k^{-1})$ represents the "connected sum of k tori," i.e. a torus with k holes.
- This type of torus can be obtained from a sphere by adding k "handles" to a sphere.



Theorem 1.1

Every surface has a polygonal representation of one of the following forms:

- AA^{-1};
- $(A_1 B_1 A_1^{-1} B_1^{-1})(A_2 B_2 A_2^{-1} B_2^{-1}) \dots (A_k B_k A_k^{-1} B_k^{-1})$;
- $(A_1 A_1)(A_2 A_2) \dots (A_k A_k)$.

- $(A_1 A_1)(A_2 A_2) \dots (A_k A_k)$ represents the "connected sum of k real projective planes."

Theorem 1.1

Every surface has a polygonal representation of one of the following forms:

- AA^{-1};
- $(A_1 B_1 A_1^{-1} B_1^{-1})(A_2 B_2 A_2^{-1} B_2^{-1}) \dots (A_k B_k A_k^{-1} B_k^{-1})$;
- $(A_1 A_1)(A_2 A_2) \dots (A_k A_k)$.

- $(A_1 A_1)(A_2 A_2) \dots (A_k A_k)$ represents the "connected sum of k real projective planes."
- We can obtain this surface by starting with a sphere, removing k disks, and then gluing a Möbius strip along the boundary of each removed disk in the sphere.

Theorem 1.1

Every surface has a polygonal representation of one of the following forms:

- AA^{-1};
- $(A_1 B_1 A_1^{-1} B_1^{-1})(A_2 B_2 A_2^{-1} B_2^{-1}) \ldots (A_k B_k A_k^{-1} B_k^{-1})$;
- $(A_1 A_1)(A_2 A_2) \ldots (A_k A_k)$.

- $(A_1 A_1)(A_2 A_2) \ldots (A_k A_k)$ represents the "connected sum of k real projective planes."
- We can obtain this surface by starting with a sphere, removing k disks, and then gluing a Möbius strip along the boundary of each removed disk in the sphere.
- Adding one Möbius strip in this way is called "adding a crosscap."

Theorem 1.1

Every surface has a polygonal representation of one of the following forms:

- AA^{-1};
- $(A_1B_1A_1^{-1}B_1^{-1})(A_2B_2A_2^{-1}B_2^{-1})\dots(A_kB_kA_k^{-1}B_k^{-1})$;
- $(A_1A_1)(A_2A_2)\dots(A_kA_k)$.

- $(A_1A_1)(A_2A_2)\dots(A_kA_k)$ represents the "connected sum of k real projective planes."
- We can obtain this surface by starting with a sphere, removing k disks, and then gluing a Möbius strip along the boundary of each removed disk in the sphere.
- Adding one Möbius strip in this way is called "adding a crosscap."
- So, $(A_1A_1)(A_2A_2)\dots(A_kA_k)$ is the surface obtained from the sphere by adding k crosscaps.

Theorem 1.1

Every surface has a polygonal representation of one of the following forms:

- AA^{-1};
- $(A_1B_1A_1^{-1}B_1^{-1})(A_2B_2A_2^{-1}B_2^{-1})\ldots(A_kB_kA_k^{-1}B_k^{-1})$;
- $(A_1A_1)(A_2A_2)\ldots(A_kA_k)$.

- Spheres and connected sums of tori are "orientable surfaces."

Theorem 1.1

Every surface has a polygonal representation of one of the following forms:

- AA^{-1};
- $(A_1B_1A_1^{-1}B_1^{-1})(A_2B_2A_2^{-1}B_2^{-1})\dots(A_kB_kA_k^{-1}B_k^{-1})$;
- $(A_1A_1)(A_2A_2)\dots(A_kA_k)$.

- Spheres and connected sums of tori are "orientable surfaces."
  - The *genus* of the sphere is zero, and the *genus* of a connected sum of k tori is k.

Theorem 1.1

Every surface has a polygonal representation of one of the following forms:

- AA^{-1};
- $(A_1 B_1 A_1^{-1} B_1^{-1})(A_2 B_2 A_2^{-1} B_2^{-1}) \dots (A_k B_k A_k^{-1} B_k^{-1})$;
- $(A_1 A_1)(A_2 A_2) \dots (A_k A_k)$.

- Spheres and connected sums of tori are "orientable surfaces."
  - The *genus* of the sphere is zero, and the *genus* of a connected sum of k tori is k.
- Connected sums of projective planes are "non-orientable surfaces."

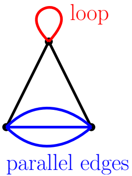
Theorem 1.1

Every surface has a polygonal representation of one of the following forms:

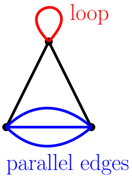
- AA^{-1};
- $(A_1 B_1 A_1^{-1} B_1^{-1})(A_2 B_2 A_2^{-1} B_2^{-1}) \dots (A_k B_k A_k^{-1} B_k^{-1})$;
- $(A_1 A_1)(A_2 A_2) \dots (A_k A_k)$.

- Spheres and connected sums of tori are "orientable surfaces."
  - The *genus* of the sphere is zero, and the *genus* of a connected sum of k tori is k.
- Connected sums of projective planes are "non-orientable surfaces."
  - The *genus* of a connected sum of k real projective planes is k.

- A “multigraph” is a graph thay may possibly have loops and parallel edges.



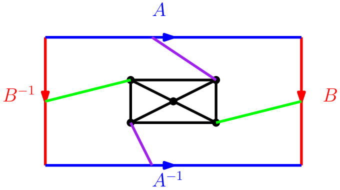
- A “multigraph” is a graph thay may possibly have loops and parallel edges.

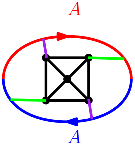


- Just as we can draw graphs (and multigraphs) on a sphere, we can draw them on any other surface.

- Generally speaking, it is more convenient to use polygonal representation for drawing, than to draw directly on the surface in question.

- Generally speaking, it is more convenient to use polygonal representation for drawing, than to draw directly on the surface in question.
- For instance, below is a drawing of K_5 on the torus (left) and on the projective plane (right). (Note how the green edge and the purple edge "wrap around" the rectangle.)





Euler polyhedral formula

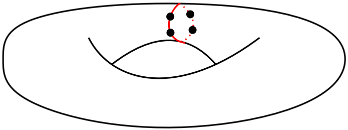
Let G be any connected planar multigraph. Then for any drawing of G on the sphere (without edge crossings), we have that

$$V - E + F = 2,$$

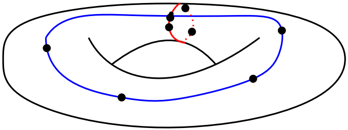
where V is the number of vertices, E the number of edges, and F the number of faces of the drawing.

Proof. Discrete Math.

- A *net* on a surface is a multigraph drawing on that surface (with no edge crossings) in which every face is homeomorphic to an open disk.

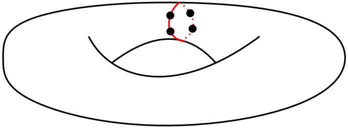


not a net

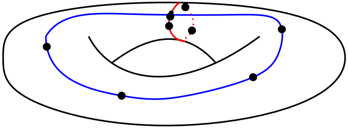


net

- A *net* on a surface is a multigraph drawing on that surface (with no edge crossings) in which every face is homeomorphic to an open disk.



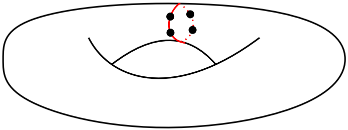
not a net



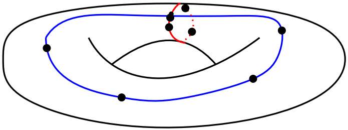
net

- Note that the net (or rather, the multigraph whose drawing it is) must be connected.

- A *net* on a surface is a multigraph drawing on that surface (with no edge crossings) in which every face is homeomorphic to an open disk.



not a net



net

- Note that the net (or rather, the multigraph whose drawing it is) must be connected.
- Our next goal is to generalize the Euler polyhedral formula for surfaces of arbitrary genus.

- Remarks:
  - If G is a net on a surface S, then subdividing an edge ℓ times does not change the value of $V - E + F$.

- Remarks:
  - If G is a net on a surface S, then subdividing an edge ℓ times does not change the value of $V - E + F$.
    - This is because both V and E increase by ℓ, and F remains unchanged.

- Remarks:
    - If G is a net on a surface S, then subdividing an edge ℓ times does not change the value of $V - E + F$.
        - This is because both V and E increase by ℓ, and F remains unchanged.
    - Further, adding an edge between two existing vertices and passing through a face does not change $V - E + F$.

- Remarks:
  - If G is a net on a surface S, then subdividing an edge ℓ times does not change the value of $V - E + F$.
    - This is because both V and E increase by ℓ, and F remains unchanged.
  - Further, adding an edge between two existing vertices and passing through a face does not change $V - E + F$.
    - This is because both E and F increase by one, and V remains unchanged.

- Remarks:
  - If G is a net on a surface S, then subdividing an edge ℓ times does not change the value of $V - E + F$.
    - This is because both V and E increase by ℓ, and F remains unchanged.
  - Further, adding an edge between two existing vertices and passing through a face does not change $V - E + F$.
    - This is because both E and F increase by one, and V remains unchanged.
    - Here, we are using the fact that each face is homeomorphic to a disk, and so adding an edge between two existing vertices necessarily “splits” an existing face into two).

Theorem 2.1

Let G be a net on an (orientable or non-orientable) surface S of genus k. Let V be the number of vertices, E the number of edges, and F the number of faces of this net. Then:

(a) if S is orientable, then $V - E + F = 2 - 2k$;

(b) if S is non-orientable, then $V - E + F = 2 - k$.

Proof.

Theorem 2.1

Let G be a net on an (orientable or non-orientable) surface S of genus k. Let V be the number of vertices, E the number of edges, and F the number of faces of this net. Then:

(a) if S is orientable, then $V - E + F = 2 - 2k$;

(b) if S is non-orientable, then $V - E + F = 2 - k$.

Proof. (a) Assume that S is orientable.

Theorem 2.1

Let G be a net on an (orientable or non-orientable) surface S of genus k. Let V be the number of vertices, E the number of edges, and F the number of faces of this net. Then:

(a) if S is orientable, then $V - E + F = 2 - 2k$;

(b) if S is non-orientable, then $V - E + F = 2 - k$.

Proof. (a) Assume that S is orientable. If $k = 0$, then S is the sphere, and we are done by the Euler polyhedral formula.

Theorem 2.1

Let G be a net on an (orientable or non-orientable) surface S of genus k. Let V be the number of vertices, E the number of edges, and F the number of faces of this net. Then:

(a) if S is orientable, then $V - E + F = 2 - 2k$;

(b) if S is non-orientable, then $V - E + F = 2 - k$.

Proof. (a) Assume that S is orientable. If $k = 0$, then S is the sphere, and we are done by the Euler polyhedral formula. So, we may assume that $k \geq 1$.

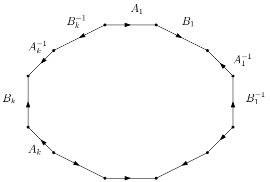
Theorem 2.1

Let G be a net on an (orientable or non-orientable) surface S of genus k. Let V be the number of vertices, E the number of edges, and F the number of faces of this net. Then:

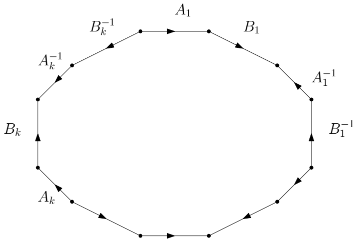
(a) if S is orientable, then $V - E + F = 2 - 2k$;

(b) if S is non-orientable, then $V - E + F = 2 - k$.

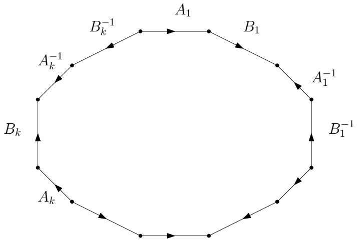
Proof. (a) Assume that S is orientable. If $k = 0$, then S is the sphere, and we are done by the Euler polyhedral formula. So, we may assume that $k \geq 1$. Then S is the connected sum of k tori, and it has a polygonal representation
$(A_1 B_1 A_1^{-1} B_1^{-1}) \dots (A_k B_k A_k^{-1} B_k^{-1})$.



Proof (continued). Reminder: S is the connected sum of k tori, represented by $(A_1 B_1 A_1^{-1} B_1^{-1}) \dots (A_k B_k A_k^{-1} B_k^{-1})$. WTS $V - E + F = 2 - 2k$.

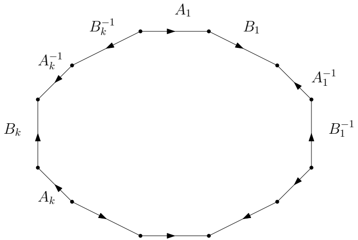


Proof (continued). Reminder: S is the connected sum of k tori, represented by $(A_1 B_1 A_1^{-1} B_1^{-1}) \dots (A_k B_k A_k^{-1} B_k^{-1})$. WTS $V - E + F = 2 - 2k$.



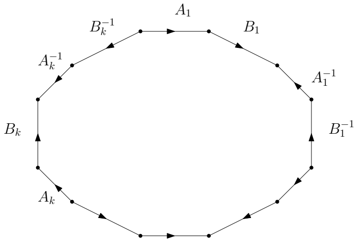
Note that the $4k$ vertices of this polygon all correspond to the same point of the surface S; we may assume that this point is a vertex of G (if not, just "move" the net a bit until it is).

Proof (continued). Reminder: S is the connected sum of k tori, represented by $(A_1B_1A_1^{-1}B_1^{-1})\dots(A_kB_kA_k^{-1}B_k^{-1})$. WTS $V - E + F = 2 - 2k$.



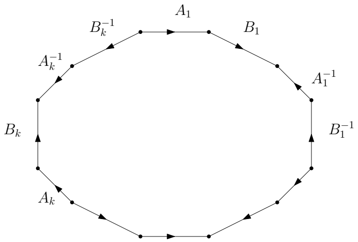
Note that the $4k$ vertices of this polygon all correspond to the same point of the surface S; we may assume that this point is a vertex of G (if not, just "move" the net a bit until it is). Next, we will assume (informal!!) that G intersects the boundary of the polygon in only finitely many points;

Proof (continued). Reminder: S is the connected sum of k tori, represented by $(A_1B_1A_1^{-1}B_1^{-1})\dots(A_kB_kA_k^{-1}B_k^{-1})$. WTS $V - E + F = 2 - 2k$.



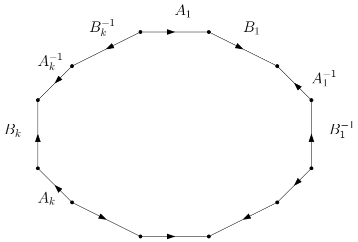
Note that the $4k$ vertices of this polygon all correspond to the same point of the surface S; we may assume that this point is a vertex of G (if not, just "move" the net a bit until it is). Next, we will assume (informal!!) that G intersects the boundary of the polygon in only finitely many points; we turn each point of intersection of the net and the polygon boundary into a vertex (this is just edge subdivision, and so $V - E + F$ does not change).

Proof (continued). Reminder: S is the connected sum of k tori, represented by $(A_1 B_1 A_1^{-1} B_1^{-1}) \dots (A_k B_k A_k^{-1} B_k^{-1})$. WTS $V - E + F = 2 - 2k$.



Finally, we turn the boundary of the polygon into edges (subdivided according to the vertices that appear on the boundary); this produces $2k$ (potentially subdivided) loops in our net, and it does not alter $V - E + F$.

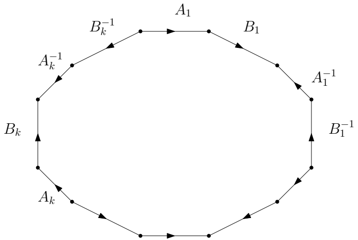
Proof (continued). Reminder: S is the connected sum of k tori, represented by $(A_1B_1A_1^{-1}B_1^{-1})\dots(A_kB_kA_k^{-1}B_k^{-1})$. WTS $V - E + F = 2 - 2k$.



Finally, we turn the boundary of the polygon into edges (subdivided according to the vertices that appear on the boundary); this produces $2k$ (potentially subdivided) loops in our net, and it does not alter $V - E + F$.

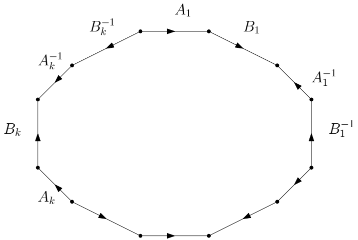
Now, our net G on the surface S can be "translated" into a plane drawing in the natural way: we simply place our polygon in the plane.

Proof (continued). Reminder: S is the connected sum of k tori, represented by $(A_1 B_1 A_1^{-1} B_1^{-1}) \dots (A_k B_k A_k^{-1} B_k^{-1})$. WTS $V - E + F = 2 - 2k$.



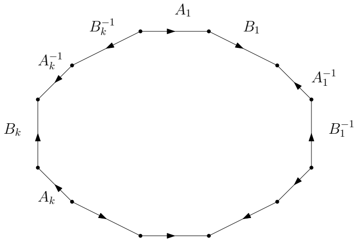
Let V_p be the number of vertices of G on S that lie in the interior of the edges of the polygon

Proof (continued). Reminder: S is the connected sum of k tori, represented by $(A_1 B_1 A_1^{-1} B_1^{-1}) \ldots (A_k B_k A_k^{-1} B_k^{-1})$. WTS $V - E + F = 2 - 2k$.



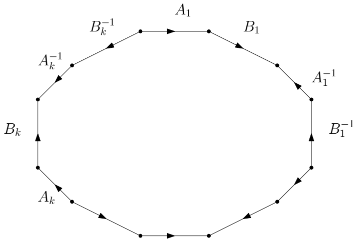
Let V_p be the number of vertices of G on S that lie in the interior of the edges of the polygon (so, in our plane drawing, this turns into $2V_p$ vertices, because each such vertex "doubles"),

Proof (continued). Reminder: S is the connected sum of k tori, represented by $(A_1B_1A_1^{-1}B_1^{-1})\dots(A_kB_kA_k^{-1}B_k^{-1})$. WTS $V - E + F = 2 - 2k$.



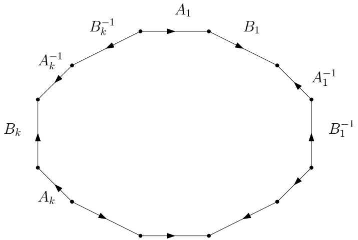
Let V_p be the number of vertices of G on S that lie in the interior of the edges of the polygon (so, in our plane drawing, this turns into $2V_p$ vertices, because each such vertex "doubles"), and let E_p be the number of edges of G on S that lie on the polygon

Proof (continued). Reminder: S is the connected sum of k tori, represented by $(A_1 B_1 A_1^{-1} B_1^{-1}) \ldots (A_k B_k A_k^{-1} B_k^{-1})$. WTS $V - E + F = 2 - 2k$.



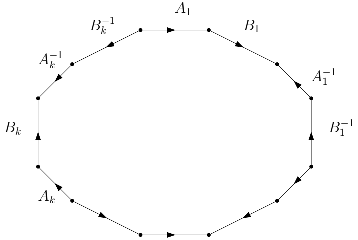
Let V_p be the number of vertices of G on S that lie in the interior of the edges of the polygon (so, in our plane drawing, this turns into $2V_p$ vertices, because each such vertex "doubles"), and let E_p be the number of edges of G on S that lie on the polygon (so, in our plane drawing, this turns into $2E_p$ edges, because each such edge "doubles").

Proof (continued). Reminder: S is the connected sum of k tori, represented by $(A_1B_1A_1^{-1}B_1^{-1})\dots(A_kB_kA_k^{-1}B_k^{-1})$. WTS $V - E + F = 2 - 2k$.



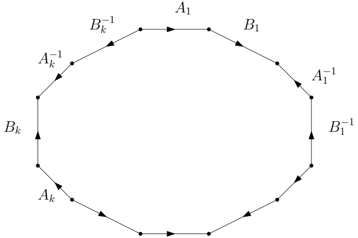
Let V_p be the number of vertices of G on S that lie in the interior of the edges of the polygon (so, in our plane drawing, this turns into $2V_p$ vertices, because each such vertex "doubles"), and let E_p be the number of edges of G on S that lie on the polygon (so, in our plane drawing, this turns into $2E_p$ edges, because each such edge "doubles"). Further, one vertex of G got turned into $4k$ vertices (the vertices of the polygon) in our plane drawing.

Proof (continued). Reminder: S is the connected sum of k tori, represented by $(A_1 B_1 A_1^{-1} B_1^{-1}) \dots (A_k B_k A_k^{-1} B_k^{-1})$. WTS $V - E + F = 2 - 2k$.



So, the plane drawing has
$4k + 2V_p + (V - 1 - V_p) = V + V_p + 4k - 1$ vertices,
$2E_p + (E - E_p) = E + E_p$ edges, and $F + 1$ faces (because of the exterior face).

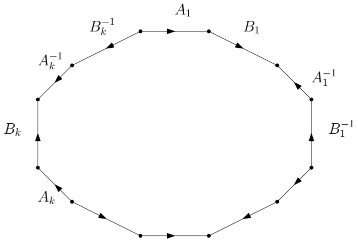
Proof (continued). Reminder: S is the connected sum of k tori, represented by $(A_1B_1A_1^{-1}B_1^{-1})\dots(A_kB_kA_k^{-1}B_k^{-1})$. WTS $V - E + F = 2 - 2k$.



Now, the Euler polyhedral formula gives us

$$(V + V_p + 4k - 1) - (E + E_p) + (F + 1) \; = \; 2,$$

Proof (continued). Reminder: S is the connected sum of k tori, represented by $(A_1 B_1 A_1^{-1} B_1^{-1}) \ldots (A_k B_k A_k^{-1} B_k^{-1})$. WTS $V - E + F = 2 - 2k$.

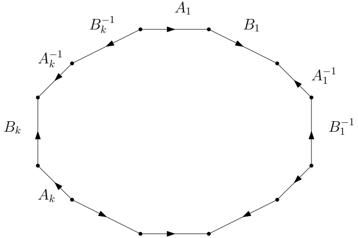


Now, the Euler polyhedral formula gives us

$$(V + V_p + 4k - 1) - (E + E_p) + (F + 1) \quad = \quad 2,$$

and therefore, $(V - E + F) + (V_p - E_p) = 2 - 4k$.

Proof (continued). Reminder: S is the connected sum of k tori, represented by $(A_1B_1A_1^{-1}B_1^{-1})\dots(A_kB_kA_k^{-1}B_k^{-1})$. WTS $V - E + F = 2 - 2k$.

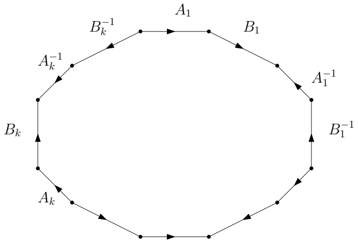


Now, the Euler polyhedral formula gives us

$$(V + V_p + 4k - 1) - (E + E_p) + (F + 1) \;=\; 2,$$

and therefore, $(V - E + F) + (V_p - E_p) = 2 - 4k$. But note that $E_p = V_p + 2k$.

Proof (continued). Reminder: S is the connected sum of k tori, represented by $(A_1 B_1 A_1^{-1} B_1^{-1}) \ldots (A_k B_k A_k^{-1} B_k^{-1})$. WTS $V - E + F = 2 - 2k$.

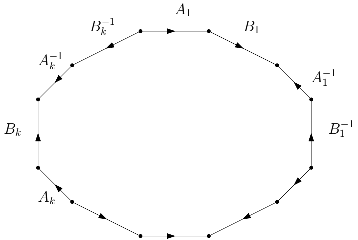


Now, the Euler polyhedral formula gives us

$$(V + V_p + 4k - 1) - (E + E_p) + (F + 1) \;\; = \;\; 2,$$

and therefore, $(V - E + F) + (V_p - E_p) = 2 - 4k$. But note that $E_p = V_p + 2k$. So, $V - E + F - 2k = 2 - 4k$,

Proof (continued). Reminder: S is the connected sum of k tori, represented by $(A_1 B_1 A_1^{-1} B_1^{-1}) \ldots (A_k B_k A_k^{-1} B_k^{-1})$. WTS $V - E + F = 2 - 2k$.



Now, the Euler polyhedral formula gives us

$$(V + V_p + 4k - 1) - (E + E_p) + (F + 1) \quad = \quad 2,$$

and therefore, $(V - E + F) + (V_p - E_p) = 2 - 4k$. But note that $E_p = V_p + 2k$. So, $V - E + F - 2k = 2 - 4k$, and therefore $V - E + F = 2 - 2k$, which is what we needed.

Theorem 2.1

Let G be a net on an (orientable or non-orientable) surface S of genus k. Let V be the number of vertices, E the number of edges, and F the number of faces of this net. Then:

(a) if S is orientable, then $V - E + F = 2 - 2k$;

(b) if S is non-orientable, then $V - E + F = 2 - k$.

Proof (continued). We have now proven (a).

Theorem 2.1

Let G be a net on an (orientable or non-orientable) surface S of genus k. Let V be the number of vertices, E the number of edges, and F the number of faces of this net. Then:

(a) if S is orientable, then $V - E + F = 2 - 2k$;

(b) if S is non-orientable, then $V - E + F = 2 - k$.

Proof (continued). We have now proven (a).

(b) Assume that S is non-orientable; then S is the connected sum of k projective planes. Let $(A_1A_1)\dots(A_kA_k)$ be the polygonal representation of the surface S. The proof is now almost identical to that of part (a), except that we have a $2k$-gon, rather than a $4k$-gon, and so the computation yields $V - E + F = 2 - k$.

Theorem 2.1

Let G be a net on an (orientable or non-orientable) surface S of genus k. Let V be the number of vertices, E the number of edges, and F the number of faces of this net. Then:

(a) if S is orientable, then $V - E + F = 2 - 2k$;

(b) if S is non-orientable, then $V - E + F = 2 - k$.

Theorem 2.1

Let G be a net on an (orientable or non-orientable) surface S of genus k. Let V be the number of vertices, E the number of edges, and F the number of faces of this net. Then:

(a) if S is orientable, then $V - E + F = 2 - 2k$;

(b) if S is non-orientable, then $V - E + F = 2 - k$.

Corollary 2.2

Let G be a multigraph drawing (with no edge crossings) on a surface S of genus k.[a] Let V be the number of vertices, E the number of edges, and F the number of faces of this net. Then:

(a) if S is orientable, then $V - E + F \geq 2 - 2k$;

(b) if S is non-orientable, then $V - E + F \geq 2 - k$.

---

[a] Note: G need not be a net, that is, it is possible that not all faces are homeomorphic to disks.

Corollary 2.2

Let G be a multigraph drawing (with no edge crossings) on a surface S of genus k. Let V be the number of vertices, E the number of edges, and F the number of faces of this net. Then:

(a) if S is orientable, then $V - E + F \geq 2 - 2k$;

(b) if S is non-orientable, then $V - E + F \geq 2 - k$.

Proof. We keep adding edges to G (without creating edge crossings) until we obtain a net. (Note that this may possibly decrease the value of $V - E + F$, but it cannot increase it.) The result is a net, and so the result follows from Theorem 2.1.

Definition

The *Euler characteristic* of a surface S, denoted by $\mathrm{ec}(S)$, is the number $V - E + F$, where V, E, and F are, respectively, the number of vertices, edges, and faces of some net on S.

Definition

The *Euler characteristic* of a surface S, denoted by $\mathrm{ec}(S)$, is the number $V - E + F$, where V, E, and F are, respectively, the number of vertices, edges, and faces of some net on S.

- By Theorem 2.1, the Euler characteristic is is well defined, i.e. the number $\mathrm{ec}(S)$ depends only on the surface S, and not on the particular choice of a net.

Definition

The *Euler characteristic* of a surface S, denoted by $\mathrm{ec}(S)$, is the number $V - E + F$, where V, E, and F are, respectively, the number of vertices, edges, and faces of some net on S.

- By Theorem 2.1, the Euler characteristic is is well defined, i.e. the number $\mathrm{ec}(S)$ depends only on the surface S, and not on the particular choice of a net.
- Theorem 2.1 states that if S is an orientable surface of genus k, then $\mathrm{ec}(S) = 2 - 2k$; on the other hand, if S is a non-orientable surface of genus S, then $\mathrm{ec}(S) = 2 - k$.

Definition

The *Euler characteristic* of a surface S, denoted by $\mathrm{ec}(S)$, is the number $V - E + F$, where V, E, and F are, respectively, the number of vertices, edges, and faces of some net on S.

- By Theorem 2.1, the Euler characteristic is is well defined, i.e. the number $\mathrm{ec}(S)$ depends only on the surface S, and not on the particular choice of a net.
- Theorem 2.1 states that if S is an orientable surface of genus k, then $\mathrm{ec}(S) = 2 - 2k$; on the other hand, if S is a non-orientable surface of genus S, then $\mathrm{ec}(S) = 2 - k$.
  - So, the Euler characteristic of the sphere is 2, and the Euler characteristic of the torus is 0. The Euler characteristic of the projective plane is 1.

Definition

The *Euler characteristic* of a surface S, denoted by $\text{ec}(S)$, is the number $V - E + F$, where V, E, and F are, respectively, the number of vertices, edges, and faces of some net on S.

- By Theorem 2.1, the Euler characteristic is is well defined, i.e. the number $\text{ec}(S)$ depends only on the surface S, and not on the particular choice of a net.
- Theorem 2.1 states that if S is an orientable surface of genus k, then $\text{ec}(S) = 2 - 2k$; on the other hand, if S is a non-orientable surface of genus S, then $\text{ec}(S) = 2 - k$.
  - So, the Euler characteristic of the sphere is 2, and the Euler characteristic of the torus is 0. The Euler characteristic of the projective plane is 1.
- Corollary 2.2 implies that if G is a multigraph drawing on a surface S, then $V - E + F \geq \text{ec}(S)$, where V, E, and F are the number of vertices, edges, and faces of the drawing.

Corollary 2.3

Let G be a (simple) graph on at least two edges, drawn on an (orientable or non-orientable) surface S (without edge crossings). Then $|E(G)| \leq 3|V(G)| - 3\mathrm{ec}(S)$. Consequently, the average degree of G is at most $6 - \frac{6\mathrm{ec}(S)}{|V(G)|}$.

Proof.

Corollary 2.3

Let G be a (simple) graph on at least two edges, drawn on an (orientable or non-orientable) surface S (without edge crossings). Then $|E(G)| \leq 3|V(G)| - 3\text{ec}(S)$. Consequently, the average degree of G is at most $6 - \frac{6\text{ec}(S)}{|V(G)|}$.

Proof. For each face f, we define $\ell(f)$ to be the number of edges incident with f, with each edge incident with f on both sides counting twice.

Corollary 2.3

Let G be a (simple) graph on at least two edges, drawn on an (orientable or non-orientable) surface S (without edge crossings). Then $|E(G)| \leq 3|V(G)| - 3\text{ec}(S)$. Consequently, the average degree of G is at most $6 - \frac{6\text{ec}(S)}{|V(G)|}$.

Proof. For each face f, we define $\ell(f)$ to be the number of edges incident with f, with each edge incident with f on both sides counting twice. Since G is simple and $|E(G)| \geq 2$, we see that $\ell(f) \geq 3$ for all faces f.

Corollary 2.3

Let G be a (simple) graph on at least two edges, drawn on an (orientable or non-orientable) surface S (without edge crossings). Then $|E(G)| \leq 3|V(G)| - 3\text{ec}(S)$. Consequently, the average degree of G is at most $6 - \frac{6\text{ec}(S)}{|V(G)|}$.

Proof. For each face f, we define $\ell(f)$ to be the number of edges incident with f, with each edge incident with f on both sides counting twice. Since G is simple and $|E(G)| \geq 2$, we see that $\ell(f) \geq 3$ for all faces f. If $F(G)$ is the set of all faces, we get

$$2|E(G)| \;=\; \sum_{f \in F(G)} \ell(f) \;\geq\; 3|F(G)|,$$

and therefore, $|F(G)| \leq \frac{2}{3}|E(G)|$.

Corollary 2.3

Let G be a (simple) graph on at least two edges, drawn on an (orientable or non-orientable) surface S (without edge crossings). Then $|E(G)| \leq 3|V(G)| - 3\text{ec}(S)$. Consequently, the average degree of G is at most $6 - \frac{6\text{ec}(S)}{|V(G)|}$.

Proof. For each face f, we define $\ell(f)$ to be the number of edges incident with f, with each edge incident with f on both sides counting twice. Since G is simple and $|E(G)| \geq 2$, we see that $\ell(f) \geq 3$ for all faces f. If $F(G)$ is the set of all faces, we get

$$2|E(G)| \;=\; \sum_{f \in F(G)} \ell(f) \;\geq\; 3|F(G)|,$$

and therefore, $|F(G)| \leq \frac{2}{3}|E(G)|$. Now we compute

$$\begin{aligned} |E(G)| &\leq |V(G)| + |F(G)| - \text{ec}(S) && \text{by Corollary 2.2} \\ &\leq |V(G)| + \tfrac{2}{3}|E(G)| - \text{ec}(S), \end{aligned}$$

Corollary 2.3

Let G be a (simple) graph on at least two edges, drawn on an (orientable or non-orientable) surface S (without edge crossings). Then $|E(G)| \leq 3|V(G)| - 3\mathrm{ec}(S)$. Consequently, the average degree of G is at most $6 - \frac{6\mathrm{ec}(S)}{|V(G)|}$.

Proof. For each face f, we define $\ell(f)$ to be the number of edges incident with f, with each edge incident with f on both sides counting twice. Since G is simple and $|E(G)| \geq 2$, we see that $\ell(f) \geq 3$ for all faces f. If $F(G)$ is the set of all faces, we get

$$2|E(G)| \;=\; \textstyle\sum_{f \in F(G)} \ell(f) \;\geq\; 3|F(G)|,$$

and therefore, $|F(G)| \leq \frac{2}{3}|E(G)|$. Now we compute

$$\begin{aligned} |E(G)| &\leq |V(G)| + |F(G)| - \mathrm{ec}(S) && \text{by Corollary 2.2} \\ &\leq |V(G)| + \tfrac{2}{3}|E(G)| - \mathrm{ec}(S), \end{aligned}$$

and so $|E(G)| \leq 3|V(G)| - 3\mathrm{ec}(S)$.

Corollary 2.3

Let G be a (simple) graph on at least two edges, drawn on an (orientable or non-orientable) surface S of genus k (without edge crossings). Then $|E(G)| \leq 3|V(G)| + 3k - 6$. Consequently, the average degree of G is at most $6 + \frac{6(k-2)}{|V(G)|}$.

Proof (continued). Reminder: $|E(G)| \leq 3|V(G)| - 3\text{ec}(S)$.

Corollary 2.3

Let G be a (simple) graph on at least two edges, drawn on an (orientable or non-orientable) surface S of genus k (without edge crossings). Then $|E(G)| \leq 3|V(G)| + 3k - 6$. Consequently, the average degree of G is at most $6 + \frac{6(k-2)}{|V(G)|}$.

Proof (continued). Reminder: $|E(G)| \leq 3|V(G)| - 3\text{ec}(S)$.

Finally, since the average degree of G is $\frac{2|E(G)|}{|V(G)|}$, the inequality above immediately implies that the average degree of G is at most $6 - \frac{6\text{ec}(S)}{|V(G)|}$.

Definition

For an integer $c \leq 2$, we define the *Heawood number* as follows:

$$H(c) \quad := \quad \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor.$$

Definition

For an integer $c \leq 2$, we define the *Heawood number* as follows:

$$H(c) \quad := \quad \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor.$$

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\mathrm{ec}(S))$.

Proof.

Definition

For an integer $c \le 2$, we define the *Heawood number* as follows:

$$H(c) \quad := \quad \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor.$$

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \le H(\mathrm{ec}(S))$.

Proof. Fix a surface S and a graph G that can be drawn on S without edge crossings. To simplify notation, set $c := \mathrm{ec}(S)$. We must show that $\chi(G) \le H(c)$.

Definition

For an integer $c \leq 2$, we define the *Heawood number* as follows:

$$H(c) \quad := \quad \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor .$$

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\text{ec}(S))$.

Proof. Fix a surface S and a graph G that can be drawn on S without edge crossings. To simplify notation, set $c := \text{ec}(S)$. We must show that $\chi(G) \leq H(c)$.

Suppose first that S is the sphere, so that G is a planar graph and $c = 2$. By the **Four Color Theorem**, G is 4-colorable. On the other hand, $H(c) = H(2) = 4$. So, $\chi(G) \leq 4 = H(c)$.

Definition

For an integer $c \leq 2$, we define the *Heawood number* as follows:

$$H(c) \quad := \quad \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor.$$

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\mathrm{ec}(S))$.

Proof. Fix a surface S and a graph G that can be drawn on S without edge crossings. To simplify notation, set $c := \mathrm{ec}(S)$. We must show that $\chi(G) \leq H(c)$.

Suppose first that S is the sphere, so that G is a planar graph and $c = 2$. By the **Four Color Theorem**, G is 4-colorable. On the other hand, $H(c) = H(2) = 4$. So, $\chi(G) \leq 4 = H(c)$.

From now on, WMA S is not a sphere, so that $c \leq 1$.

- Reminder: $H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\mathrm{ec}(S))$.

Proof (continued). Reminder: $c \leq 1$.

- Reminder: $H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\mathrm{ec}(S))$.

Proof (continued). Reminder: $c \leq 1$.

Suppose that there exists a graph G that can be drawn on S without edge crossings, but satisfies $\chi(G) > H(c)$;

- Reminder: $H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\mathrm{ec}(S))$.

Proof (continued). Reminder: $c \leq 1$.

Suppose that there exists a graph G that can be drawn on S without edge crossings, but satisfies $\chi(G) > H(c)$; WMA G was chosen with as few vertices as possible.

- Reminder: $H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\mathrm{ec}(S))$.

Proof (continued). Reminder: $c \leq 1$.

Suppose that there exists a graph G that can be drawn on S without edge crossings, but satisfies $\chi(G) > H(c)$; WMA G was chosen with as few vertices as possible. Set $n := |V(G)|$; clearly, $n \geq \chi(G) \geq H(c) + 1$.

- Reminder: $H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\text{ec}(S))$.

Proof (continued). Reminder: $c \leq 1$.

Suppose that there exists a graph G that can be drawn on S without edge crossings, but satisfies $\chi(G) > H(c)$; WMA G was chosen with as few vertices as possible. Set $n := |V(G)|$; clearly, $n \geq \chi(G) \geq H(c) + 1$. Moreover, $\delta(G) \geq H(c)$, for otherwise, we fix a vertex $v \in V(G)$ such that $d_G(v) \leq H(c) - 1$, we color $G \setminus v$ with $H(c)$ colors (this is possible by the minimality of n), and then we extend this coloring to a proper coloring of G using at most $H(c)$ colors by assigning to v a color not used on any of its neighbors.

- Reminder: $H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\text{ec}(S))$.

Proof (continued). Reminder: $c \leq 1$.

Suppose that there exists a graph G that can be drawn on S without edge crossings, but satisfies $\chi(G) > H(c)$; WMA G was chosen with as few vertices as possible. Set $n := |V(G)|$; clearly, $n \geq \chi(G) \geq H(c) + 1$. Moreover, $\delta(G) \geq H(c)$, for otherwise, we fix a vertex $v \in V(G)$ such that $d_G(v) \leq H(c) - 1$, we color $G \setminus v$ with $H(c)$ colors (this is possible by the minimality of n), and then we extend this coloring to a proper coloring of G using at most $H(c)$ colors by assigning to v a color not used on any of its neighbors. On the other hand, by Corollary 2.3, the average degree in G is at most $6 - \frac{6c}{n}$.

- Reminder: $H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\mathrm{ec}(S))$.

Proof (continued). Reminder: $c \leq 1$.

Suppose that there exists a graph G that can be drawn on S without edge crossings, but satisfies $\chi(G) > H(c)$; WMA G was chosen with as few vertices as possible. Set $n := |V(G)|$; clearly, $n \geq \chi(G) \geq H(c) + 1$. Moreover, $\delta(G) \geq H(c)$, for otherwise, we fix a vertex $v \in V(G)$ such that $d_G(v) \leq H(c) - 1$, we color $G \setminus v$ with $H(c)$ colors (this is possible by the minimality of n), and then we extend this coloring to a proper coloring of G using at most $H(c)$ colors by assigning to v a color not used on any of its neighbors. On the other hand, by Corollary 2.3, the average degree in G is at most $6 - \frac{6c}{n}$. So, $H(c) \leq 6 - \frac{6c}{n}$.

- Reminder: $H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\mathrm{ec}(S))$.

Proof (continued). Reminder: $c \leq 1$ and $H(c) \leq 6 - \frac{6c}{n}$.

- Reminder: $H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\mathrm{ec}(S))$.

Proof (continued). Reminder: $c \leq 1$ and $H(c) \leq 6 - \frac{6c}{n}$.

Since $H(1) = 6$, the inequality above does not hold if $c = 1$.

- Reminder: $H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\mathrm{ec}(S))$.

Proof (continued). Reminder: $c \leq 1$ and $H(c) \leq 6 - \frac{6c}{n}$.
Since $H(1) = 6$, the inequality above does not hold if $c = 1$. So, $c \leq 0$.

- Reminder: $H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\mathrm{ec}(S))$.

Proof (continued). Reminder: $c \leq 1$ and $H(c) \leq 6 - \frac{6c}{n}$.

Since $H(1) = 6$, the inequality above does not hold if $c = 1$. So, $c \leq 0$. Since $n \geq H(c) + 1 > 0$, it follows that $-\frac{6c}{n} \leq -\frac{6c}{H(c)+1}$, and consequently,

$$H(c) \leq 6 - \frac{6c}{H(c)+1}.$$

- Reminder: $H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\mathrm{ec}(S))$.

Proof (continued). Reminder: $c \leq 1$ and $H(c) \leq 6 - \frac{6c}{n}$.
Since $H(1) = 6$, the inequality above does not hold if $c = 1$. So, $c \leq 0$. Since $n \geq H(c) + 1 > 0$, it follows that $-\frac{6c}{n} \leq -\frac{6c}{H(c)+1}$, and consequently,

$$H(c) \leq 6 - \frac{6c}{H(c)+1}.$$

Since $H(c) > 0$, the above is equivalent to

$$H(c)^2 - 5H(c) + 6(c-1) \leq 0.$$

- Reminder: $H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\mathrm{ec}(S))$.

Proof (continued). Reminder: $H(c)^2 - 5H(c) + 6(c-1) \leq 0$.

- Reminder: $H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\mathrm{ec}(S))$.

Proof (continued). Reminder: $H(c)^2 - 5H(c) + 6(c-1) \leq 0$.

By solving the corresponding quadratic equation, we now get that

$$\frac{5-\sqrt{49-24c}}{2} \quad \leq \quad H(c) \quad \leq \quad \frac{5+\sqrt{49-24c}}{2}.$$

- Reminder: $H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\mathrm{ec}(S))$.

Proof (continued). Reminder: $H(c)^2 - 5H(c) + 6(c-1) \leq 0$.

By solving the corresponding quadratic equation, we now get that

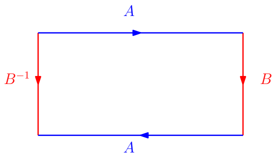
$$\frac{5-\sqrt{49-24c}}{2} \leq H(c) \leq \frac{5+\sqrt{49-24c}}{2}.$$

But from the definition of $H(c)$, we have that

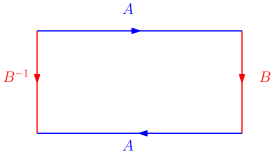
$$H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor > \frac{7+\sqrt{49-24c}}{2} - 1 = \frac{5+\sqrt{49-24c}}{2},$$

a contradiction.

- The *Klein bottle* is the surface with polygonal representation $AABB$ or $ABAB^{-1}$.

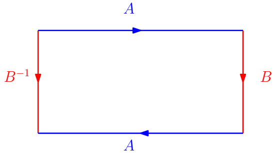


- The *Klein bottle* is the surface with polygonal representation $AABB$ or $ABAB^{-1}$.



- The Klein bottle is non-orientable, has genus 2, and Euler characteristic 0.

- The *Klein bottle* is the surface with polygonal representation $AABB$ or $ABAB^{-1}$.



- The Klein bottle is non-orientable, has genus 2, and Euler characteristic 0.

Theorem 3.2 [Ringel and Youngs]

If S is a surface other than the Klein bottle, then the complete graph $K_{H(\mathrm{ec}(S))}$ can be drawn on S (without edge crossings).

Proof. Omitted.

- Reminder: $H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\mathrm{ec}(S))$.

Theorem 3.2 [Ringel and Youngs]

If S is a surface other than the Klein bottle, then the complete graph $K_{H(\mathrm{ec}(S))}$ can be drawn on S (without edge crossings).

- Reminder: $H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\mathrm{ec}(S))$.

Theorem 3.2 [Ringel and Youngs]

If S is a surface other than the Klein bottle, then the complete graph $K_{H(\mathrm{ec}(S))}$ can be drawn on S (without edge crossings).

- Theorem 3.2 proves that the bound from Theorem 3.1 is best possible, except when S is the Klein bottle.

- Reminder: $H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\mathrm{ec}(S))$.

Theorem 3.2 [Ringel and Youngs]

If S is a surface other than the Klein bottle, then the complete graph $K_{H(\mathrm{ec}(S))}$ can be drawn on S (without edge crossings).

- Theorem 3.2 proves that the bound from Theorem 3.1 is best possible, except when S is the Klein bottle.
- For the Klein bottle, we can get a better bound.

- Reminder: $H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\mathrm{ec}(S))$.

Theorem 3.2 [Ringel and Youngs]

If S is a surface other than the Klein bottle, then the complete graph $K_{H(\mathrm{ec}(S))}$ can be drawn on S (without edge crossings).

- Theorem 3.2 proves that the bound from Theorem 3.1 is best possible, except when S is the Klein bottle.
- For the Klein bottle, we can get a better bound.
- Recall that the Euler characteristic of the Klein bottle is 0, and note that $H(0) = 7$.

- Reminder: $H(c) = \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor$.

Theorem 3.1

If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(\text{ec}(S))$.

Theorem 3.2 [Ringel and Youngs]

If S is a surface other than the Klein bottle, then the complete graph $K_{H(\text{ec}(S))}$ can be drawn on S (without edge crossings).

- Theorem 3.2 proves that the bound from Theorem 3.1 is best possible, except when S is the Klein bottle.
- For the Klein bottle, we can get a better bound.
- Recall that the Euler characteristic of the Klein bottle is 0, and note that $H(0) = 7$.
- However, as we shall see, the maximum chromatic number of the Klein bottle is 6 (see Theorem 3.4).

Lemma 3.3

K_6 can be drawn on the Klein bottle (without edge crossings), but K_7 cannot.

Proof. Omitted.

Lemma 3.3

K_6 can be drawn on the Klein bottle (without edge crossings), but K_7 cannot.

Proof. Omitted.

Brooks' theorem

Let G be a connected graph that is neither a complete graph nor an odd cycle. Then $\chi(G) \leq \Delta(G)$.

Proof. Next time.

Lemma 3.3

K_6 can be drawn on the Klein bottle (without edge crossings), but K_7 cannot.

Proof. Omitted.

Brooks' theorem

Let G be a connected graph that is neither a complete graph nor an odd cycle. Then $\chi(G) \leq \Delta(G)$.

Proof. Next time.

Theorem 3.4

Let G be a graph that can be drawn on the Klein bottle (without edge crossings). Then $\chi(G) \leq 6$.

Theorem 3.4

Let G be a graph that can be drawn on the Klein bottle (without edge crossings). Then $\chi(G) \leq 6$.

Proof.

Theorem 3.4

Let G be a graph that can be drawn on the Klein bottle (without edge crossings). Then $\chi(G) \leq 6$.

Proof. Suppose otherwise, i.e. suppose $\chi(G) \geq 7$.

Theorem 3.4

Let G be a graph that can be drawn on the Klein bottle (without edge crossings). Then $\chi(G) \leq 6$.

Proof. Suppose otherwise, i.e. suppose $\chi(G) \geq 7$. We may assume that, among all graphs that can be drawn on the Klein bottle but are not 6-colorable, G has the smallest possible number of vertices.

Theorem 3.4

Let G be a graph that can be drawn on the Klein bottle (without edge crossings). Then $\chi(G) \leq 6$.

Proof. Suppose otherwise, i.e. suppose $\chi(G) \geq 7$. We may assume that, among all graphs that can be drawn on the Klein bottle but are not 6-colorable, G has the smallest possible number of vertices. Note that this means that $\delta(G) \geq 6$.

Theorem 3.4

Let G be a graph that can be drawn on the Klein bottle (without edge crossings). Then $\chi(G) \leq 6$.

Proof. Suppose otherwise, i.e. suppose $\chi(G) \geq 7$. We may assume that, among all graphs that can be drawn on the Klein bottle but are not 6-colorable, G has the smallest possible number of vertices. Note that this means that $\delta(G) \geq 6$. On the other hand, since the Klein bottle has Euler characteristic 0, Corollary 2.3 guarantees that G has average degree at most 6.

Theorem 3.4

Let G be a graph that can be drawn on the Klein bottle (without edge crossings). Then $\chi(G) \leq 6$.

Proof. Suppose otherwise, i.e. suppose $\chi(G) \geq 7$. We may assume that, among all graphs that can be drawn on the Klein bottle but are not 6-colorable, G has the smallest possible number of vertices. Note that this means that $\delta(G) \geq 6$. On the other hand, since the Klein bottle has Euler characteristic 0, Corollary 2.3 guarantees that G has average degree at most 6. But this is possible only if G is 6-regular.

Theorem 3.4

Let G be a graph that can be drawn on the Klein bottle (without edge crossings). Then $\chi(G) \leq 6$.

Proof. Suppose otherwise, i.e. suppose $\chi(G) \geq 7$. We may assume that, among all graphs that can be drawn on the Klein bottle but are not 6-colorable, G has the smallest possible number of vertices. Note that this means that $\delta(G) \geq 6$. On the other hand, since the Klein bottle has Euler characteristic 0, Corollary 2.3 guarantees that G has average degree at most 6. But this is possible only if G is 6-regular. Now, by the minimality of $|V(G)|$, we know that G is connected.

Theorem 3.4

Let G be a graph that can be drawn on the Klein bottle (without edge crossings). Then $\chi(G) \leq 6$.

Proof. Suppose otherwise, i.e. suppose $\chi(G) \geq 7$. We may assume that, among all graphs that can be drawn on the Klein bottle but are not 6-colorable, G has the smallest possible number of vertices. Note that this means that $\delta(G) \geq 6$. On the other hand, since the Klein bottle has Euler characteristic 0, Corollary 2.3 guarantees that G has average degree at most 6. But this is possible only if G is 6-regular. Now, by the minimality of $|V(G)|$, we know that G is connected. Since $\chi(G) \geq 7$, Brooks' theorem guarantees that $G \cong K_7$.

Theorem 3.4

Let G be a graph that can be drawn on the Klein bottle (without edge crossings). Then $\chi(G) \leq 6$.

Proof. Suppose otherwise, i.e. suppose $\chi(G) \geq 7$. We may assume that, among all graphs that can be drawn on the Klein bottle but are not 6-colorable, G has the smallest possible number of vertices. Note that this means that $\delta(G) \geq 6$. On the other hand, since the Klein bottle has Euler characteristic 0, Corollary 2.3 guarantees that G has average degree at most 6. But this is possible only if G is 6-regular. Now, by the minimality of $|V(G)|$, we know that G is connected. Since $\chi(G) \geq 7$, Brooks' theorem guarantees that $G \cong K_7$. But this contradicts Lemma 3.3.