NDMI012: Combinatorics and Graph Theory 2

Lecture #5
Graphs on surfaces

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1 Surfaces

A *surface* is a connected 2-dimensional compact manifold with no boundary. This definition contains several terms we have not defined, and whose formal definition we omit. Here is an intuitive explanation:

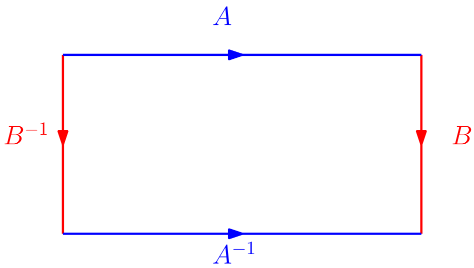
- "2-dimensional manifold with no boundary" means that each point has a neighborhood homeomorphic to an open disk;
- "compact" means that that the surface admits a triangulation with finitely many triangles;
- "connected" means that there is just one piece.

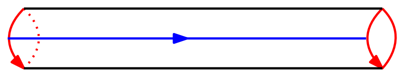
The sphere and the torus are surfaces. However, the plane is not a surface (because it is not compact). A closed disk is not a surface, either, since it has a boundary.

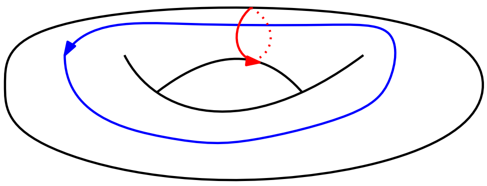
In what follows, we consider two surfaces to be the "same" if they are "homeomorphic," that is, if there is a bijection f between them such that both f and f^{-1} are continuous. So, if we can obtain one surface from the other by "stretching," then the two surfaces are the same. Thus, a tetrahedron is simply a sphere for our purposes, but a torus is not a sphere.

Here is one way of forming a torus: we start with a rectangle (see the picture below), and then we identify the two (directed) blue edges and two (directed) red edges. Importantly, the corresponding edges must be identified in the direction represented by the arrows. (In the picture below, we first identify the blue edges to get a "tube," and then we identify the two red edges/circles to get a torus. In our picture, the four vertices of the rectangle all get identified to the same point on the torus.) Note that the blue edges are labeled A (for clockwise direction) and A^{-1} (for counterclockwise direction);

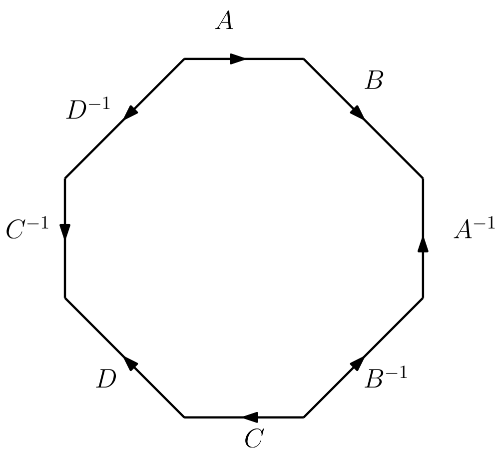
a similar labeling applies to B and B^{-1}. Symbolically, the rectangle is represented by the string $ABA^{-1}B^{-1}$.





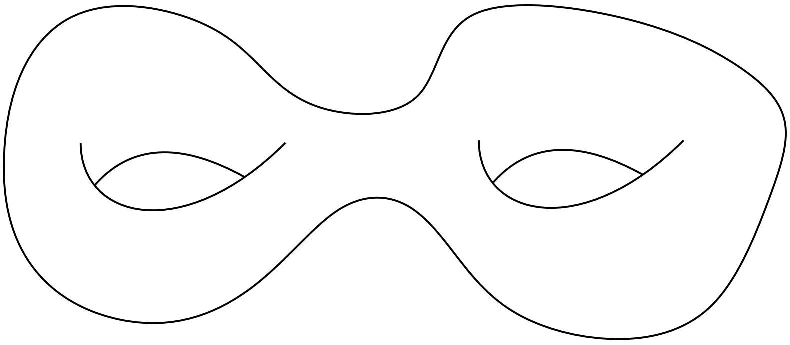


If we identify corresponding edges in the octagon $ABA^{-1}B^{-1}CDC^{-1}D^{-1}$ below, then we get a double torus (also called the "connected sum of two tori"), as you can check.[1]



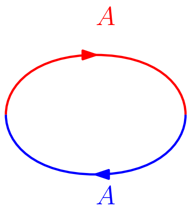
$ABA^{-1}B^{-1}CDC^{-1}D^{-1}$

[1]Alternatively, you can watch watch this video: https://www.youtube.com/watch?v=G1yyfPShgqw

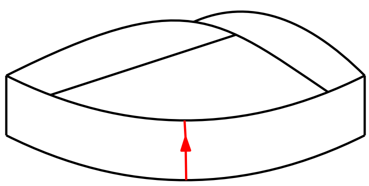


double torus

The *real projective plane* (or simply *projective plane*) is the surface obtained by starting from the sphere, and then identifying each pair of antipodal points. The projective plane has polygonal representation AA (see below). Here, we have two A's (red and blue); they are to be identified in the direction indicated.



Unlike the torus, the projective plane cannot be embedded in $\mathbb{R}^3$. Still, there is a geometric interpretation. Take a rectangle shown below on the left (think of it as a piece of paper), twist it, and identify the two vertical edges (as shown by the arrows). The result (on the bottom right) is called the *Möbius strip*. Note that the boundary of the Möbius strip consists of just one circle, and the Möbius strip has just one "side."

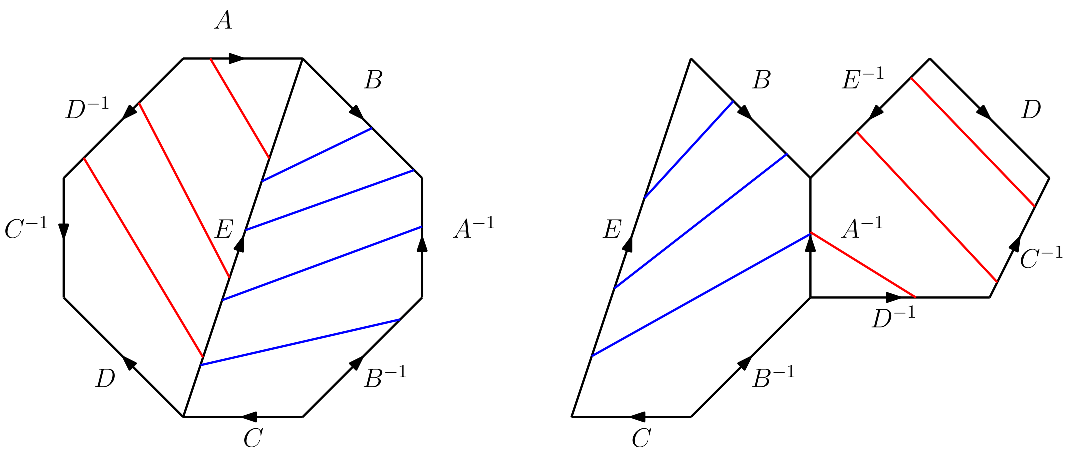


Now, take a sphere, cut out a small disk from it, and then glue the Möbius strip along the boundary obtained by removing the disk. The result is the projective plane (the circle of the Möbius strip corresponds to the edge A from our AA representation of the projective plane). Equivalently, if we cut out a disk from the projective plane, we obtain the Möbius strip.[2]

[2]It is not necessarily obvious that these three descriptions (sphere with antipodal points identified; polygonal AA representation; and Möbius strip with a disk) of the projective plane are equivalent, i.e. that they yield the same surface. For an animation that explains this, see this video: https://www.youtube.com/watch?v=u0VkikpElMo

The projective plane is a type of "non-orientable" surface, which roughly means that we cannot set up a left-right distinction. Intuitively, imagine a two-dimensional bug on the surface of the Möbius strip (which is part of the projective plane). If the bug keeps going forward, it will eventually come back to the same place (and facing in the same direction), but with left and right reversed. This sort of thing is impossible on "orientable" surfaces such as the sphere or the torus (or double torus, triple torus, etc.).

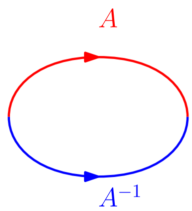
Now, any $2n$-gon, with edges labeled $A_1, \ldots, A_n$ (in any order), with each letter appearing exactly twice on the $2n$-gon, either in the form A (for clockwise direction) or A^{-1} (for counterclockwise direction) can be transformed into a surface via gluing using the rules described above.[3] Some labellings are equivalent. For example, the two octagons below obviously "encode" the same surface (i.e. after gluing, we get the same surface, in this case, the double torus). We first "cut" and then "glue" the polygon on the left in order to obtain the polygon on the right.[4]



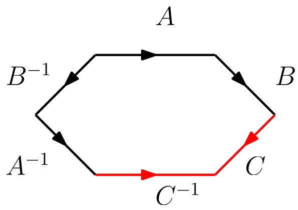
$ABA^{-1}B^{-1}CDC^{-1}D^{-1}$ $\qquad$ $BE^{-1}DC^{-1}D^{-1}B^{-1}CE$

Sometimes, we have "unnecessary" letters/edges in our polygon, as the picture below shows. Both polygons represent a torus.

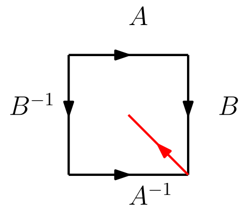
[3]Note: AA^{-1} is simply the sphere.



[4]Note that the red pentagons have different shapes. This is due to the drawing software, but this is not important: we are allowed to "strech" and "shrink" any way we like.



$ABCC^{-1}A^{-1}B^{-1}$



$ABA^{-1}B^{-1}$

An argument resembling the one indicated by the pictures above yields the following "classification theorem" (whose proof we omit).

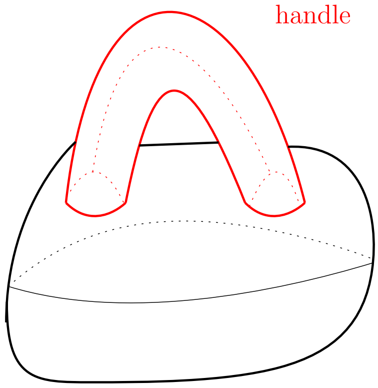
Theorem 1.1. *Every surface has a polygonal representation of one of the following forms:*

- AA^{-1};
- $(A_1B_1A_1^{-1}B_1^{-1})(A_2B_2A_2^{-1}B_2^{-1})\dots(A_kB_kA_k^{-1}B_k^{-1})$;
- $(A_1A_1)(A_2A_2)\dots(A_kA_k)$.

Importantly, Theorem 1.1 does **not** state that each polygonal representation of a surface has one of the forms from the theorem. As a matter of fact, each surface has infinitely many polygonal representations.[5] Theorem 1.1 merely states that for each surface S, one of its representations has a "canonical" from (i.e. one of the forms from the theorem).

We remark that the surface with polygonal representation AA^{-1} is simply the sphere. Further, the surface having a polygonal representation $(A_1B_1A_1^{-1}B_1^{-1})(A_2B_2A_2^{-1}B_2^{-1})\dots(A_kB_kA_k^{-1}B_k^{-1})$ is the "connected sum of k tori," i.e. a torus with k holes. This type of torus can be obtained from a sphere by adding k "handles" to a sphere. Adding a handle to a surface we have already constructed consists of removing two small disks from the surface, and then connecting them via a "handle" (a tube). Spheres and connected sums of tori are "orientable surfaces." The *genus* of the sphere is zero, and the *genus* of a connected sum of k tori is k.

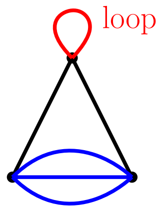
[5] Indeed, for any polygonal representation of a surface, we can obtain another polygonal representation by adding AA^{-1} (where is a "new" letter) to the end. We can repeat the procedure arbitrarily many times, thus creating infinitely many polygonal representaitons of the same surface.



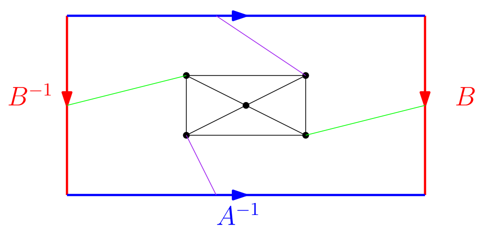
Finally, $(A_1A_1)(A_2A_2)\dots(A_kA_k)$ represents the "connected sum of k real projective planes." We can obtain this surface by starting with a sphere, removing k disks, and then gluing a Möbius strip along the boundary of each removed disk in the sphere. Adding one Möbius strip in this way is called "adding a crosscap." So, $(A_1A_1)(A_2A_2)\dots(A_kA_k)$ is the surface obtained from the sphere by adding k crosscaps. Connected sums of projective planes are "non-orientable surfaces." The *genus* of a connected sum of k real projective planes is k.

2 Graph drawing on surfaces

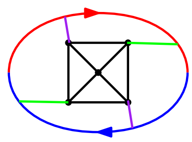
A "multigraph" is a graph thay may possibly have loops and parallel edges.



Just as we can draw graphs (and multigraphs) on a sphere, we can draw them on any other surface. Generally speaking, it is more convenient to use polygonal representation for drawing, than to draw directly on the surface in question. For instance, below is a drawing of K_5 on the torus. (Note how the green edge and the purple edge "wrap around" the rectangle.)



Further, below is a drawing of K_5 on the projective plane (once again, note how the green edge and the purple edge wrap around).



The following was proven in Discrete Math.[6]

Euler polyhedral formula. *Let G be any connected planar multigraph. Then for any drawing of G on the sphere (without edge crossings), we have that*

$$V - E + F \quad = \quad 2,$$

where V is the number of vertices, E the number of edges, and F the number of faces of the drawing.

A *net* on a surface is a multigraph drawing on that surface (with no edge crossings) in which every face is homeomorphic to an open disk. (Note that the net (or rather, the multigraph whose drawing it is) must be connected.) Our next theorem is a generalization of the Euler polyhedral formula for surfaces of arbitrary genus. Before stating and proving the theorem, we make an observation. If G is a net on a surface S, then subdividing an edge ℓ times does not change the value of $V - E + F$ (because both V and E increase by ℓ, and F remains unchanged). Further, adding an edge between two existing vertices and passing through a face does not change $V - E + F$ (because both E and F increase by one, and V remains unchanged; here, we are using the fact that each face is homeomorphic to a disk, and so adding an edge between two existing vertices necessarily "splits" an existing face into two).

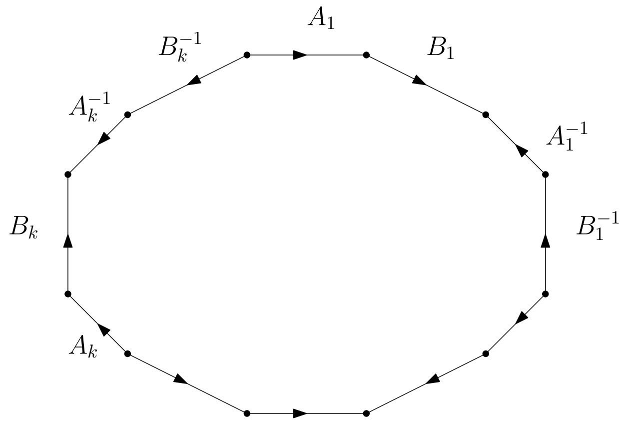
Theorem 2.1. *Let G be a net on an (orientable or non-orientable) surface S of genus k. Let V be the number of vertices, E the number of edges, and F the number of faces of this net. Then:*

(a) if S is orientable, then $V - E + F = 2 - 2k$;

(b) if S is non-orientable, then $V - E + F = 2 - k$.

[6]Perhaps you saw the proof of the Euler polyhedral formula only for graphs (not multigraphs). But note that any multigraph can be turned into a graph by edge subdivision, which does not alter the expression $V - E + F$ (because subdividing an edge once increases both the number of vertices and the number of edges by one, while leaving the number of faces unchanged). So, we can easily derive the multigraph version of the Euler polyhedral formula from the graph version.

Proof. (a) Assume that S is orientable. If $k = 0$, then S is the sphere, and we are done by the Euler polyhedral formula. So, we may assume that $k \geq 1$. Then S is the connected sum of k tori, and it has a polygonal representation $(A_1 B_1 A_1^{-1} B_1^{-1}) \dots (A_k B_k A_k^{-1} B_k^{-1})$.



Note that the $4k$ vertices of this polygon all correspond to the same point of the surface S; we may assume that this point is a vertex of G (if not, just "move" the net a bit until it is). Next, we will assume that G intersects the boundary of the polygon in only finitely many points;[7] we turn each point of intersection of the net and the polygon boundary into a vertex (this is just edge subdivision, and so $V - E + F$ does not change). Finally, we turn the boundary of the polygon into edges (subdivided according to the vertices that appear on the boundary); this produces $2k$ (potentially subdivided) loops in our net,[8] and it does not alter $V - E + F$. We still call the resulting net G, and we let V be the number of vertices, E the number of edges, and F the number of faces of the net.[9]

Now, our net G on the surface S can be "translated" into a plane drawing in the natural way: we simply place our polygon in the plane. Let V_p be the number of vertices of G on S that lie in the interior of the edges of the polygon (so, in our plane drawing, this turns into $2V_p$ vertices, because each such vertex "doubles"), and let E_p be the number of edges of G on S that lie on the polygon (so, in our plane drawing, this turns into $2E_p$ edges, because each such edge "doubles"). Further, one vertex of G got turned into $4k$ vertices (the vertices of the polygon) in our plane drawing. So, the plane drawing has $4k + 2V_p + (V - 1 - V_p) = V + V_p + 4k - 1$ vertices,

[7] This part is a bit informal: a full justification of our assumption requires somewhat sophisticated topology.

[8] We have $4k$ edges on the boundary of the polygon, but after identification, they turn into $2k$ loops.

[9] Technically, we have produced a new net G', with corresponding (new) V', E', and F', and we have that $V' - E' + F' = V - E + F$. However, for the sake of notational simplicity, we just write G, V, E, F instead. Importantly, $V - E + F$ has not changed.

$2E_p + (E - E_p) = E + E_p$ edges, and $F + 1$ faces (because of the exterior face). Now, the Euler polyhedral formula gives us

$$(V + V_p + 4k - 1) - (E + E_p) + (F + 1) \quad = \quad 2,$$

and therefore,

$$(V - E + F) + (V_p - E_p) \quad = \quad 2 - 4k.$$

But note that $E_p = V_p + 2k$.[10] So, $V - E + F - 2k = 2 - 4k$, and therefore $V - E + F = 2 - 2k$, which is what we needed.

(b) Assume that S is non-orientable; then S is the connected sum of k projective planes. Let $(A_1A_1)\dots(A_kA_k)$ be the polygonal representation of the surface S. The proof is now almost identical to that of part (a), except that we have a $2k$-gon, rather than a $4k$-gon, and so the computation yields $V - E + F = 2 - k$.[11] □

Corollary 2.2. *Let G be a multigraph drawing (with no edge crossings) on a surface S of genus k.*[12] *Let V be the number of vertices, E the number of edges, and F the number of faces of this net. Then:*

(a) if S is orientable, then $V - E + F \geq 2 - 2k$;

(b) if S is non-orientable, then $V - E + F \geq 2 - k$.

Proof. We keep adding edges (possibly loops) to G without creating edge crossings, until we obtain a net.[13] The result is a net, and so the result follows from Theorem 2.1. □

The *Euler characteristic* of a surface S, denoted by $\mathrm{ec}(S)$, is the number $V - E + F$, where V, E, and F are, respectively, the number of vertices, edges, and faces of some net on S.[14] By Theorem 2.1, this is well defined, i.e. the number $\mathrm{ec}(S)$ depends only on the surface S, and not on the particular choice of a net. Moreover, Theorem 2.1 states that if S is an orientable surface of genus k, then $\mathrm{ec}(S) = 2 - 2k$; on the other hand, if S is a non-orientable surface of genus k, then $\mathrm{ec}(S) = 2 - k$.[15] Moreover, Corollary 2.2 implies that

[10] Indeed, the polygon has $4k$ edges, which correspond to $2k$ loops on the surface S. Each time we subdivide an edge, we increase both the number of vertices and the number of edges by one.

[11] Check the details!

[12] Note: G need not be a net, that is, it is possible that not all faces are homeomorphic to disks.

[13] Note that this may possibly decrease the value of $V - E + F$, but it cannot increase it.

[14] The Euler characteristic of a surface S is usually denoted by $\chi(S)$. However, some texts use $\mathrm{ec}(S)$, to avoid confusion with the chromatic number. Here, we will use the notation $\mathrm{ec}(S)$.

[15] So, the Euler characteristic of the sphere is 2, and the Euler characteristic of the torus is 0. The Euler characteristic of the projective plane is 1.

if G is a multigraph drawing on a surface S, then $V - E + F \geq \text{ec}(S)$, where V, E, and F are the number of vertices, edges, and faces of the drawing.

Note that the following corollary only holds for graphs (and not for multigraphs). For emphasis, graphs without loops and parallel edges are referred to as simple graphs.[16]

Corollary 2.3. *Let G be a (simple) graph on at least two edges, drawn on an (orientable or non-orientable) surface S (without edge crossings). Then*

$$|E(G)| \quad \leq \quad 3|V(G)| - 3ec(S).$$

Consequently, the average degree of G is at most $6 - \frac{6ec(S)}{|V(G)|}$.

Proof. For each face f, we define $\ell(f)$ to be the number of edges incident with f, with each edge incident with f on both sides counting twice. Since G is simple and $|E(G)| \geq 2$, we see that $\ell(f) \geq 3$ for all faces f. If $F(G)$ is the set of all faces, we get

$$2|E(G)| \quad = \quad \sum_{f \in F(G)} \ell(f) \quad \geq \quad 3|F(G)|,$$

and therefore,

$$|F(G)| \quad \leq \quad \tfrac{2}{3}|E(G)|$$

Now we compute

$$\begin{aligned} |E(G)| \quad &\leq \quad |V(G)| + |F(G)| - \text{ec}(S) \qquad \text{by Corollary 2.2} \\ &\leq \quad |V(G)| + \tfrac{2}{3}|E(G)| - \text{ec}(S), \end{aligned}$$

and so

$$|E(G)| \quad \leq \quad 3|V(G)| - 3\text{ec}(S).$$

Finally, since the average degree of G is $\frac{2|E(G)|}{|V(G)|}$, the inequality above immediately implies that the average degree of G is at most $6 - \frac{6\text{ec}(S)}{|V(G)|}$. □

3 The Heawood number

For an integer $c \leq 2$, we define the *Heawood number* as follows:

$$H(c) \quad := \quad \left\lfloor \frac{7+\sqrt{49-24c}}{2} \right\rfloor.$$

We remark that for the case when S is a sphere, the proof of our next theorem uses the (highly non-trivial) Four Color Theorem, which states that every planar graph is 4-colorable, i.e. has chromatic number at most four. If S is any other surface, then the proof is relatively elementary.

[16]In some texts, graphs are always assumed to be simple (i.e. loopless and without parallel edges), and in others, they are allowed to have loops and/or parallel edges.

Theorem 3.1. *If a (simple) graph G can be drawn without edge crossings on an (orientable or non-orientable) surface S, then $\chi(G) \leq H(ec(S))$.*

Proof. Fix a surface S and a graph G that can be drawn on S without edge crossings. To simplify notation, set $c := \text{ec}(S)$. We must show that $\chi(G) \leq H(c)$.

Suppose first that S is the sphere, so that G is a planar graph and $c = 2$. By the Four Color Theorem, G is 4-colorable. On the other hand, $H(c) = H(2) = 4$. So, $\chi(G) \leq 4 = H(c)$.

From now on, we may assume that S is not a sphere, so that $c \leq 1$. Suppose that there exists a graph G that can be drawn on S without edge crossings, but satisfies $\chi(G) > H(c)$; we may assume that G was chosen with as few vertices as possible. Set $n := |V(G)|$; clearly, $n \geq \chi(G) \geq H(c) + 1$. Moreover, $\delta(G) \geq H(c)$,[17] for otherwise, we fix a vertex $v \in V(G)$ such that $d_G(v) \leq H(c) - 1$, we color $G \setminus v$ with $H(c)$ colors (this is possible by the minimality of n), and then we extend this coloring to a proper coloring of G using at most $H(c)$ colors by assigning to v a color not used on any of its neighbors.[18] On the other hand, by Corollary 2.3, the average degree in G is at most $6 - \frac{6c}{n}$. So,

$$H(c) \quad \leq \quad 6 - \tfrac{6c}{n}.$$

Since $H(1) = 6$, the inequality above does not hold if $c = 1$. So, $c \leq 0$. Since $n \geq H(c) + 1 > 0$, it follows that $-\frac{6c}{n} \leq -\frac{6c}{H(c)+1}$, and consequently,

$$H(c) \quad \leq \quad 6 - \tfrac{6c}{H(c)+1}.$$

Since $H(c) > 0$, the above is equivalent to

$$H(c)^2 - 5H(c) + 6(c-1) \quad \leq \quad 0.$$

By solving the corresponding quadratic equation, we now get that

$$\tfrac{5-\sqrt{49-24c}}{2} \quad \leq \quad H(c) \quad \leq \quad \tfrac{5+\sqrt{49-24c}}{2}.$$

But from the definition of $H(c)$, we have that

$$H(c) \quad = \quad \left\lfloor \tfrac{7+\sqrt{49-24c}}{2} \right\rfloor \quad > \quad \tfrac{7+\sqrt{49-24c}}{2} - 1 \quad = \quad \tfrac{5+\sqrt{49-24c}}{2},$$

a contradiction. □

The *Klein bottle* is the surface with polygonal representation $AABB$ or $ABAB^{-1}$.[19] Note that the Klein bottle is non-orientable (and therefore cannot be embedded in $\mathbb{R}^3$);[20] it has genus 2 and Euler characteristic 0.

[17] As usual, $\delta(G)$ is the minimum degree of G, i.e. $\delta(G) := \min\{d_G(v) \mid v \in V(G)\}$.

[18] This contradicts our assumption that $\chi(G) > H(c)$.

[19] Check that these are equivalent!

[20] However, a geometric representation of the Klein bottle is possible in $\mathbb{R}^3$, provided we allow the surface to intersect itself. The key is to remember that the intersection is not "really" there, but is simply a feature of our attempt to represent the surface in $\mathbb{R}^3$. For a video, see here: https://www.youtube.com/watch?v=yaeyNjUPVqs

Theorem 3.2 (Ringel and Youngs)**.** *If S is a surface other than the Klein bottle, then the complete graph $K_{H(ec(S))}$ can be drawn on S (without edge crossings).*

We omit the proof of Theorem 3.2. Note, however, that Theorem 3.2 proves that the bound established in Theorem 3.1 is best possible, except when S is the Klein bottle. For the Klein bottle, we can get a better bound. Recall that the Euler characteristic of the Klein bottle is 0, and note that $H(0) = 7$. However, as we shall see, the maximum chromatic number of a graph that can be drawn on the Klein bottle without edge crossings is 6 (see Theorem 3.4). We begin with the following Lemma, whose proof we omit.[21]

Lemma 3.3. *K_6 can be drawn on the Klein bottle (without edge crossings), but K_7 cannot.*

We will also need Brooks' theorem (stated below), whose proof will be given in Lecture 6. As usual, $\Delta(G)$ is the maximum degree of a graph G, i.e. $\Delta(G) := \max\{d_G(v) \mid v \in V(G)\}$.

Brooks' theorem. *Let G be a connected graph that is neither a complete graph nor an odd cycle. Then $\chi(G) \leq \Delta(G)$.*

Theorem 3.4. *Let G be a graph that can be drawn on the Klein bottle (without edge crossings). Then $\chi(G) \leq 6$.*

Proof. Suppose otherwise, i.e. suppose $\chi(G) \geq 7$. We may assume that, among all graphs that can be drawn on the Klein bottle but are not 6-colorable, G has the smallest possible number of vertices. Note that this means that $\delta(G) \geq 6$.[22] On the other hand, since the Klein bottle has Euler characteristic 0, Corollary 2.3 guarantees that G has average degree at most 6. But this is possible only if G is 6-regular. Now, by the minimality of $|V(G)|$, we know that G is connected.[23] Since $\chi(G) \geq 7$, Brooks' theorem guarantees that $G \cong K_7$. But this contradicts Lemma 3.3. □

[21] You can try to prove the lemma as an exercise. Showing that K_6 can be drawn on the Klein bottle is easy, but showing that K_7 cannot be drawn in such a way is more complicated.

[22] Indeed, suppose G has a vertex v of degree at most 5. Then $G \setminus v$ is 6-colorable (by the minimality of $|V(G)|$). We then fix a proper coloring of G with at most six colors, and we extend it to a proper coloring of G with at most six colors by assigning to v a color not used on any of its neighbors. This contradicts our assumption that $\chi(G) \geq 7$.

[23] Otherwise, we take H to be a component of G such that $\chi(H) = \chi(G)$, and we observe that H contradicts the minimality of $|V(G)|$.