NDMI012: Combinatorics and Graph Theory 2

Lecture #4

Minors and planar graphs (part II)

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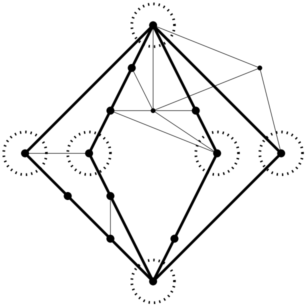
Definition

A graph H is a *topological minor* of a graph G, and we write $H \preceq_t G$, if G contains some subdivision of H as a subgraph. The vertices of this subdivision that correspond to the vertices of H are called *branch vertices*.

- The graph below contains $K_{2,4}$ as a topological minor.

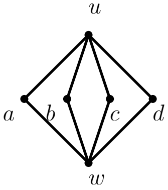


Definition

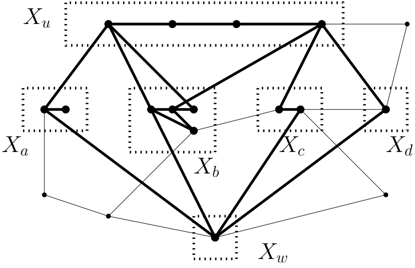
A graph H is a *minor* of a graph G, and we write $H \preceq_m G$, if there exists a family $\{X_v\}_{v \in V(H)}$ of pairwise disjoint, non-empty subsets of $V(G)$, called *branch sets*, such that

- $G[X_v]$ is connected for all $v \in V(H)$, and
- for all $uv \in E(H)$, there is an edge between X_u and X_v in G.

- For example, the graph below (on the right) contains $K_{2,4}$ as a minor.



$K_{2,4}$



Kuratowski's theorem [Kuratowski, 1930; Wagner, 1937]

Let G be a graph. Then the following are equivalent:

(a) G is planar;

(b) G contains neither K_5 nor $K_{3,3}$ as a minor;

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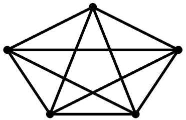
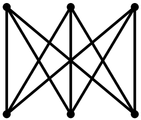


K_5



$K_{3,3}$

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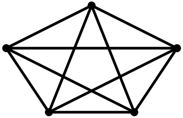

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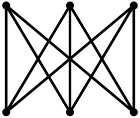

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- We proved "(a) $\Longrightarrow$ (b)" and "(b) $\Longleftrightarrow$ (c)" in the previous lecture.

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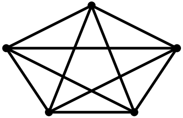

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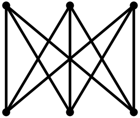
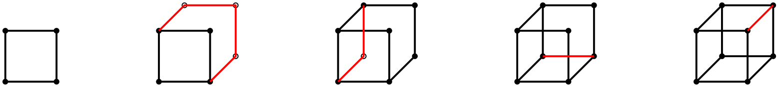


$K_{3,3}$

- We proved "(a) $\Longrightarrow$ (b)" and "(b) $\Longleftrightarrow$ (c)" in the previous lecture.
- In this lecture, we prove "(b) $\Longrightarrow$ (a)."

- A *path addition* (sometimes called *ear addition* or *open ear addition*) to a graph H is the addition to H of a path between two distinct vertices of H in such a way that no internal vertex and no edge of the path belongs to H.



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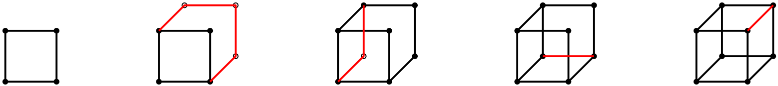

The Ear Lemma

A graph is 2-connected if and only if it is a cycle or can be obtained from a cycle by repeated path addition.

Proof. Combinatorics & Graph Theory 1.

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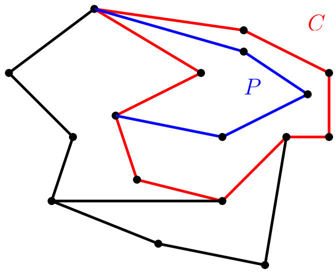
Lemma 1.2

For any plane drawing of a planar 2-connected graph G, the boundary of each face is a cycle of G.

Proof. Lecture Notes (using the Ear Lemma).



Lemma 1.3

Let G be a 3-connected graph that contains neither K_5 nor $K_{3,3}$ as a minor. Then G is planar.

Proof (outline).

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Proof (outline). We may assume inductively that the lemma is true for graphs on fewer than $|V(G)|$ vertices, that is, that for all 3-connected graphs H with $|V(H)| < |V(G)|$ and $K_5, K_{3,3} \not\preceq_m H$, we have that H is planar.

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Since G is 3-connected, we know that either $G \cong K_4$ or $|V(G)| > 4$.

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Since G is 3-connected, we know that either $G \cong K_4$ or $|V(G)| > 4$. If $G \cong K_4$, then it is clear that G is planar, and we are done. So assume that $|V(G)| > 4$.

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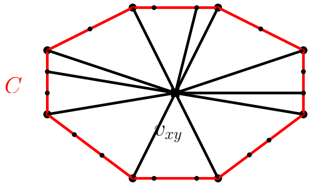
Since G is 3-connected, we know that either $G \cong K_4$ or $|V(G)| > 4$. If $G \cong K_4$, then it is clear that G is planar, and we are done. So assume that $|V(G)| > 4$. Then Lemma 1.2 from Lecture Notes 3 guarantees that G has an edge xy such that $H := G/xy$ is 3-connected. Then $H \preceq_m G$, and so $K_5, K_{3,3} \not\preceq_m H$. Now H is a 3-connected graph on $|V(G)| - 1$ vetrices, with $K_5, K_{3,3} \not\preceq_m H$; so, by the induction hypothesis, H is planar.

Lemma 1.3

Let G be a 3-connected graph that contains neither K_5 nor $K_{3,3}$ as a minor. Then G is planar.

Proof (outline, continued). Fix a plane drawing of H. If we erase v_{xy} and all the edges incident in it, we obtain a plane drawing of $H \setminus v_{xy}$. Now, let f be the face of this drawing of $H \setminus v_{xy}$ such that v_{xy} is in the interior of f. Since H is 3-connected, $H \setminus v_{xy}$ is 2-connected; so, by Lemma 1.3, the boundary of f is a cycle of $H \setminus v_{xy}$, say C. (Note that C is also a cycle of H and of G.)



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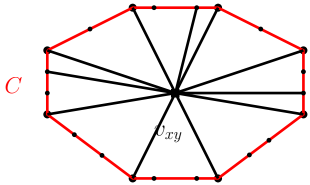

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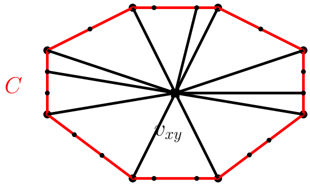

Then $N_H(v_{xy}) \subseteq V(C)$, and consequently, $N_G(x) \subseteq \{y\} \cup V(C)$ and $N_G(y) \subseteq \{x\} \cup V(C)$.

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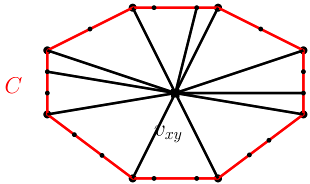

Then $N_H(v_{xy}) \subseteq V(C)$, and consequently, $N_G(x) \subseteq \{y\} \cup V(C)$ and $N_G(y) \subseteq \{x\} \cup V(C)$. Since G is 3-connected, we know that $\delta(G) \geq 3$, and in particular, $d_G(x) \geq 3$; so, since $N_G(x) \subseteq \{y\} \cup V(C)$, x has at least two neighbors in $V(C)$.

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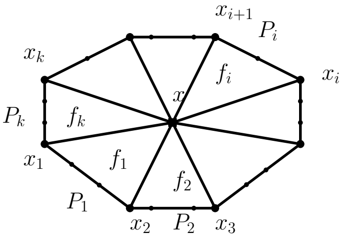
Proof (outline, continued). Let $x_1, \ldots, x_k$ be the neighbors of x in $V(C)$, listed in cyclical order (along the cycle C). For each $i \in \{1, \ldots, k\}$, let P_i be the path from x_i to x_{i+1} (we consider $x_{k+1} = x_1$) along C.



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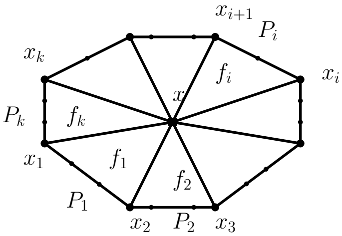

If for some $i \in \{1, \ldots, k\}$, we have that $N_G(y) \subseteq \{x\} \cup V(P_i)$, then G is planar (details: Lecture Notes).

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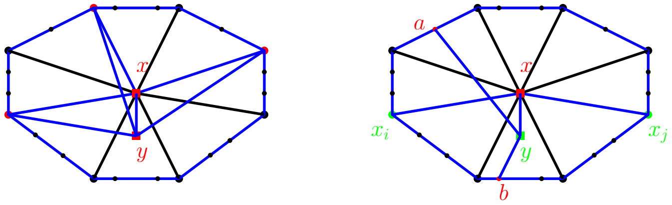
Proof (outline, continued). So, suppose that for all $i \in \{1, \ldots, k\}$, we have that $N_G(y) \not\subseteq \{x\} \cup V(P_i)$. Then either x and y have three common neighbors in $V(C)$, or y has two neighbors $a, b \in V(C)$ that are separated in C by two neighbors of x, say x_i and x_j.



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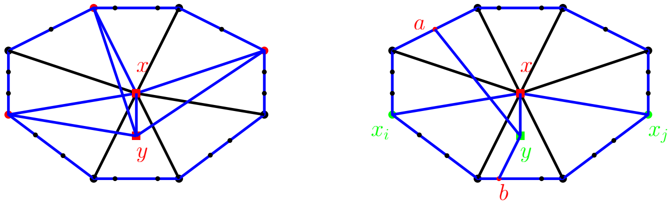

But then G contains K_5 or $K_{3,3}$ as a topological minor, a contradiction.

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Proof (outline). We may assume inductively that for all graphs H on fewer than $|V(G)|$ vertices, if $K_5, K_{3,3} \not\preceq_m H$, then H is planar.

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If $|V(G)| \leq 3$, then it is clear that G is planar. From now on, we assume that $|V(G)| \geq 4$.

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If $|V(G)| \leq 3$, then it is clear that G is planar. From now on, we assume that $|V(G)| \geq 4$.

Suppose first that G is disconnected, and let $G_1, \ldots, G_t$ be the components of G. Then by the induction hypothesis, $G_1, \ldots, G_t$ are all planar. We obtain a plane drawing of G by drawing $G_1, \ldots, G_t$ in the plane side by side.

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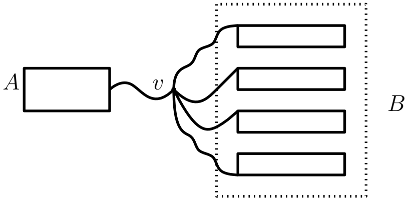
Proof (outline, continued). Next, suppose that G is connected, but not 2-connected. Then there exists a vertex $v \in V(G)$ such that $G \setminus v$ is disconnected. Let A be the vertex set of one component of $G \setminus v$, and let $B := V(G) \setminus (A \cup \{v\})$. Set $G_A := G[A \cup \{v\}]$ and $G_B := G[B \cup \{v\}]$.



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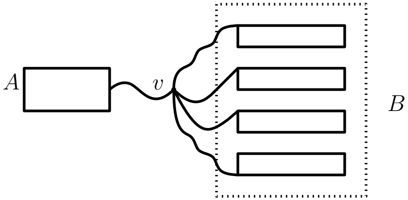

By the induction hypothesis, G_A and G_B are both planar.

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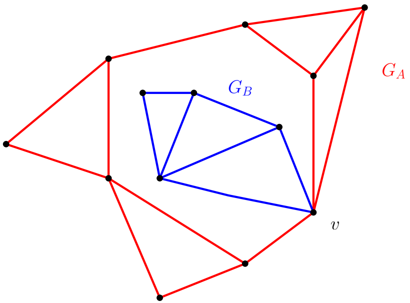
Let G be a graph that contains neither K_5 nor $K_{3,3}$ as a minor. Then G is planar.

Proof (outline, continued). We draw G_A in the plane without any edge crossings, and we let f be some face of this drawing such that v lies on the boundary of f. We then draw G_B inside f, with v coinciding in the drawing of G_A and G_B.



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- This completes the proof of Kuratowski's theorem.

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- This completes the proof of Kuratowski's theorem.

Kuratowski's theorem [Kuratowski, 1930; Wagner, 1937]

Let G be a graph. Then the following are equivalent:

(a) G is planar;

(b) G contains neither K_5 nor $K_{3,3}$ as a minor;

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For every positive integer k, every graph of chromatic number at least k contains K_k as a topological minor.

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- For $k = 3$, we observe that if a graph G satisfies $\chi(G) \geq 3$, then G is not a forest, and in particular, G contains a cycle. Every cycle is a subdivision of K_3, i.e. every cycle contains K_3 as a topological minor. So, if $\chi(G) \geq 3$, then $K_3 \preceq_t G$.

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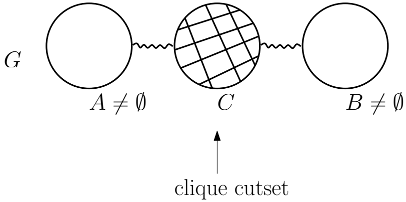
- Hajós' Conjecture is also true for $k = 4$, as we now show.
    - But first, we need a definition and a lemma.

Definition

A *clique-cutset* of a graph G is a clique $C \subsetneq V(G)$ of G such that $G \setminus C$ is disconnected.[a]

[a] In particular, if G is disconnected, then $\emptyset$ is a clique-cutset of G.



Lemma 2.1

Let G be a graph, and let C be a clique-cutset of G. Let $A_1, \ldots, A_t$ be the vertex sets of the components of $G \setminus C$. Then $\chi(G) = \max\{\chi(G[A_1 \cup C]), \ldots, \chi(G[A_t \cup C])\}$.

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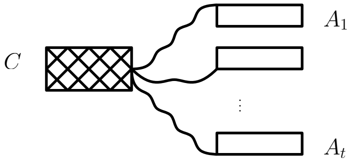

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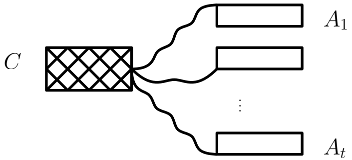

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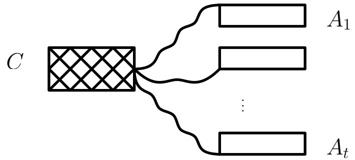

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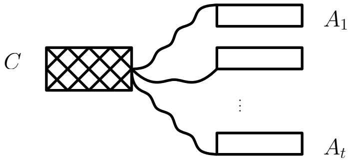

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Proof (outline, continued).

Claim 1. *G does not admit a clique-cutset. Furthermore, G is 2-connected.*

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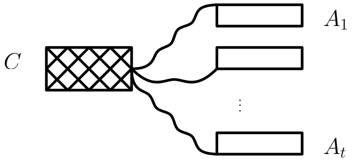
Proof of Claim 1 (outline). This follows from Lemma 2.1.



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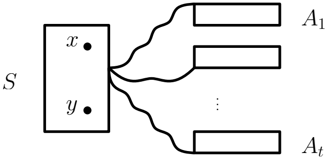

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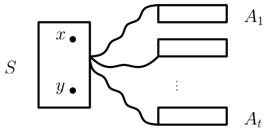

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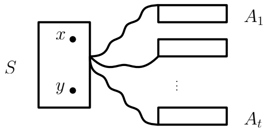

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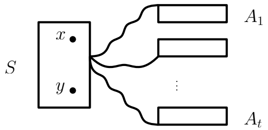

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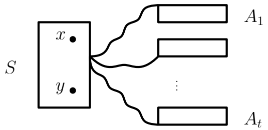

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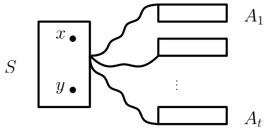

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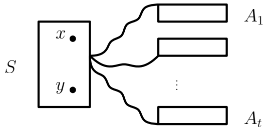

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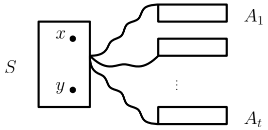

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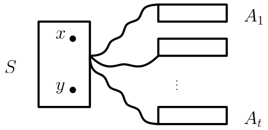

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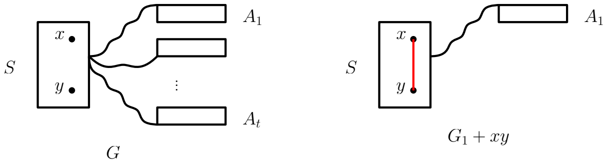

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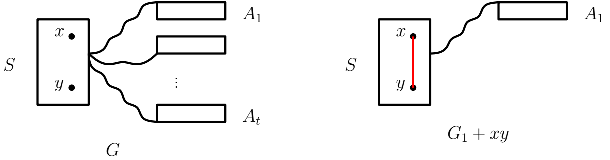

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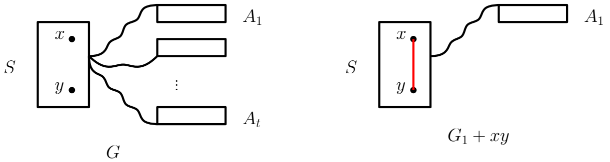

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So, we may assume that some two neighbors (call them u_1 and u_2) of u are non-adjacent.

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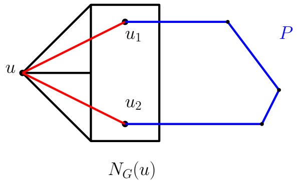

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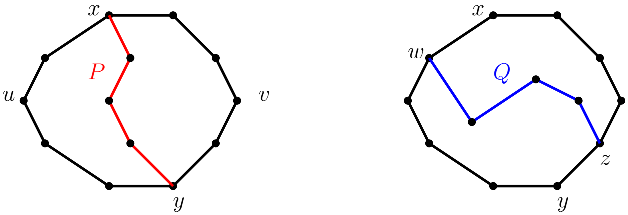

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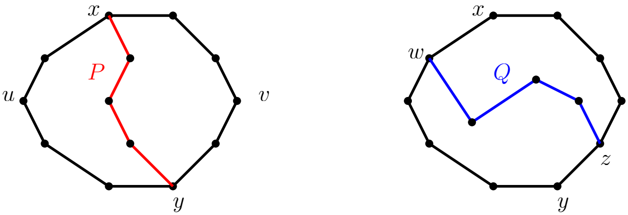

P and Q either do or do not intersect.

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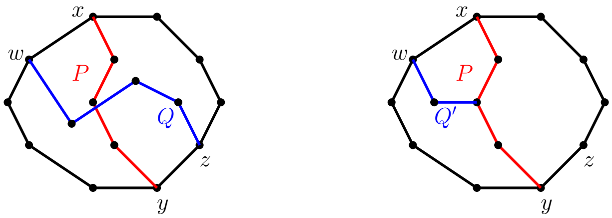

Hajós' Conjecture

For every positive integer k, every graph of chromatic number at least k contains K_k as a topological minor.

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- We now know that Hajós' Conjecture is true for $k \leq 4$.
- But for $k = 7$, it is false!

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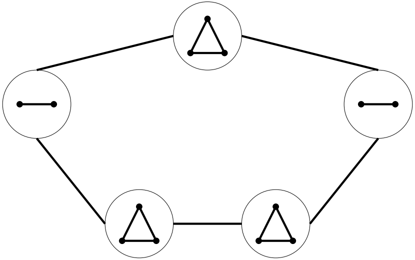

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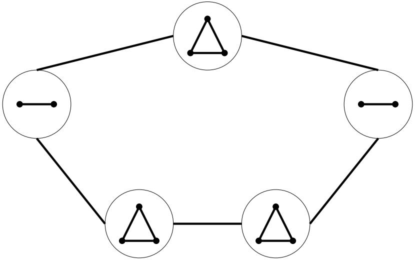

- The graph above has chromatic number 7, and yet it does not contain K_7 as a topological minor.
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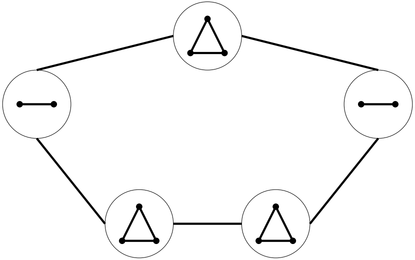

- The graph above has chromatic number 7, and yet it does not contain K_7 as a topological minor.
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- Hajós' Conjecture is open for $k = 5$ and $k = 6$.

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- Since a topological minor is a special case of a minor, Hadwiger's Conjecture is weaker than Hajós' Conjecture. Thus, since Hajós' Conjecture is true for $k \leq 4$, Hadwiger's conjecture is also true for $k \leq 4$.

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- For $k \geq 7$, the conjecture remains open.