NDMI012: Combinatorics and Graph Theory 2

Lecture #4
Minors and planar graphs (part II)

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1 Kuratowski's theorem

We stated the following theorem (usually referred to as "Kuratowski's theorem," or sometimes as the "Kuratowski-Wagner theorem") in Lecture Notes 3.

Theorem 1.1 (Kuratowski, 1930; Wagner, 1937)**.** *Let G be a graph. Then the following are equivalent:*

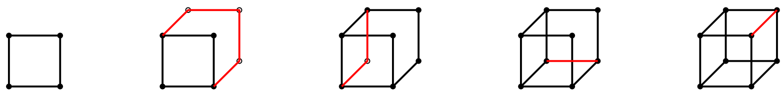
(a) G is planar;

(b) G contains neither K_5 nor $K_{3,3}$ as a minor;

(c) G contains neither K_5 nor $K_{3,3}$ as a topological minor.

We have already proven the "easy" part of Kuratowski's theorem: (a) implies (b) by Lemma 3.2 from Lecture Notes 3, and (b) is equivalent to (c) by Lemma 2.5 from Lecture Notes 3. It remains to prove the "hard" part: (b) implies (a).

A *path addition* (sometimes called *ear addition* or *open ear addition*) to a graph H is the addition to H of a path between two distinct vertices of H in such a way that no internal vertex and no edge of the path belongs to H. In the picture below, we show how the cube graph can be constructed by starting with a cycle of length four and then repeatedly adding paths (the path/ear added at each step is in red).



The following was proven in Combinatorics and Graph Theory 1.

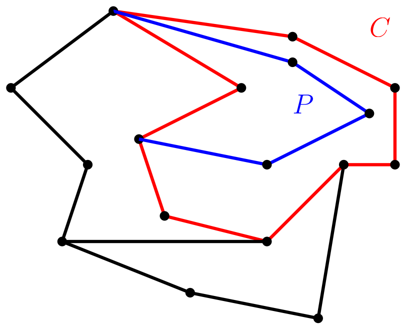
The Ear lemma. *A graph is 2-connected if and only if it is a cycle or can be obtained from a cycle by repeated path addition.*

A *plane drawing* of a planar graph is a drawing of that graph in the plane without any edge crossings.

Lemma 1.2. *For any plane drawing of a planar 2-connected graph G, the boundary of each face is a cycle of G.*

Proof. We proceed by induction on the number of edges. Let G be a planar 2-connected graph, and assume inductively that for all planar 2-connected graphs H such that $|E(H)| < |E(G)|$, in any plane drawing of H, the boundary of each face is a cycle of H.

Now, fix a plane drawing of G. If G is a cycle, then the drawing has two faces, and they are both bounded by the cycle G.[1] Suppose now that G is not a cycle. Then the Ear Lemma guarantees that G can be obtained from a 2-connected graph H by adding a path P. If we erase all the edges and all the internal vertices of P from our drawing of G, we obtain a plane drawing of H; by the induction hypothesis, each face of this drawing is bounded by a cycle of H.



We now put P back into our drawing. The path P must pass through one face of our drawing of H, and it splits this face up into two, each bounded by a cycle of G, and the other faces have unchanged boundaries. This completes the proof. $\square$

We now prove the "(b) $\implies$ (a)" part of Kuratowski's theorem for the case when G is 3-connected.

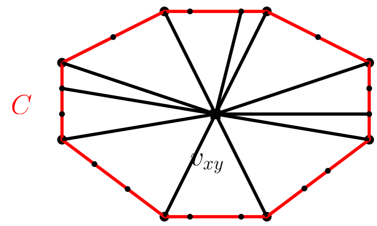
Lemma 1.3. *Let G be a 3-connected graph that contains neither K_5 nor $K_{3,3}$ as a minor. Then G is planar.*

Proof. We may assume inductively that the lemma is true for graphs on fewer than $|V(G)|$ vertices, that is, that for all 3-connected graphs H with $|V(H)| < |V(G)|$ and $K_5, K_{3,3} \not\preceq_m H$, we have that H is planar.

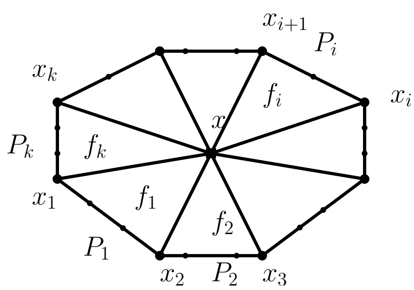
[1] Actually, this is somewhat informal. The formal proof requires a theorem from topology called the "Jordan Curve Theorem." We omit the details.

Since G is 3-connected, we know that either $G \cong K_4$ or $|V(G)| > 4$.[2] If $G \cong K_4$, then it is clear that G is planar, and we are done. So assume that $|V(G)| > 4$. Then Lemma 1.2 from Lecture Notes 3 guarantees that G has an edge xy such that $H := G/xy$ is 3-connected. By Lemma 2.1 from Lecture Notes 3, we know that $H \preceq_m G$; since $K_5, K_{3,3} \not\preceq_m G$, Lemma 2.2 from Lecture Notes 3 guarantees that $K_5, K_{3,3} \not\preceq_m H$. Now H is a 3-connected graph on $|V(G)| - 1$ vetrices, with $K_5, K_{3,3} \not\preceq_m H$; so, by the induction hypothesis, H is planar.

Fix a plane drawing of H. If we erase v_{xy} and all the edges incident in it, we obtain a plane drawing of $H \setminus v_{xy}$. Now, let f be the face of this drawing of $H \setminus v_{xy}$ such that v_{xy} is in the interior of f. Since H is 3-connected, $H \setminus v_{xy}$ is 2-connected; so, by Lemma 1.2, the boundary of f is a cycle of $H \setminus v_{xy}$, say C. (Note that C is also a cycle of H and of G.)



Then $N_H(v_{xy}) \subseteq V(C)$, and consequently, $N_G(x) \subseteq \{y\} \cup V(C)$ and $N_G(y) \subseteq \{x\} \cup V(C)$. Since G is 3-connected, we know that $\delta(G) \geq 3$, and in particular, $d_G(x) \geq 3$; so, since $N_G(x) \subseteq \{y\} \cup V(C)$, x has at least two neighbors in $V(C)$. Let $x_1, \dots, x_k$ be the neighbors of x in $V(C)$, listed in cyclical order (along the cycle C). For each $i \in \{1, \dots, k\}$, let P_i be the path from x_i to x_{i+1} (we consider $x_{k+1} = x_1$) along C, as in the picture below.

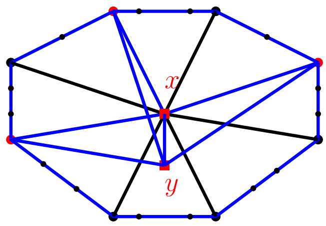


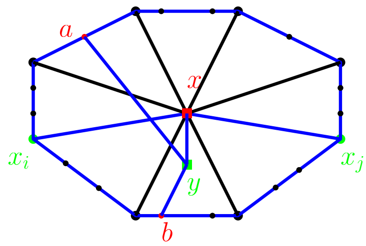
We now draw $G \setminus y$ in the plane without any edge crossings, as follows. We begin with our drawing of $H = G/xy$, we relabel v_{xy} as x, and we erase the edges between x and $V(C)$ that do not belong to $E(G)$. For each $i \in \{1, \dots, k\}$, let f_i be the face whose boundary is $x, x_i - P_i - x_{i+1}, x$ and that lies inside f. Our goal is to show that this drawing can be extended to G. If for some $i \in \{1, \dots, k\}$, we have that $N_G(y) \subseteq \{x\} \cup V(P_i)$, then we

[2] Indeed, since G is 3-connected, we know that $|V(G)| \geq 4$, and clearly, K_4 is (up to isomorphism) the only 3-connected graph on four vertices.

simply place the vertex y inside the face f_i, and we draw the edge xy as well as the edges between y and its neighbors in $V(P_i)$, and we obtain a plane drawing of G.

So, suppose that for all $i \in \{1, \dots, k\}$, we have that $N_G(y) \not\subseteq \{x\} \cup V(P_i)$. Then either x and y have three common neighbors in $V(C)$ (see the picture below, on the left), or y has two neighbors $a, b \in V(C)$ that are separated in C by two neighbors of x, say x_i and x_j (see the picture below, on the right).





In the former case, G contains K_5 as a topological minor (with x, y, and their three common neighbors in C as branch vertices), contrary to the fact that $K_5 \not\preceq_m G$.[3] In the latter case, $G[\{x, y\} \cup V(C)]$ contains $K_{3,3}$ as a topological minor, with x, y, a, b, x_i, x_j as the branch vertices,[4] contrary to the fact that $K_{3,3} \not\preceq_m G$.[5] □

Lemma 1.4. *Let G be a graph that contains neither K_5 nor $K_{3,3}$ as a minor. Then G is planar.*

Proof. We may assume inductively that for all graphs H on fewer than $|V(G)|$ vertices, if $K_5, K_{3,3} \not\preceq_m H$, then H is planar.

If $|V(G)| \leq 3$, then it is clear that G is planar. From now on, we assume that $|V(G)| \geq 4$.

Suppose first that G is disconnected, and let $G_1, \dots, G_t$ be the components of G. Then by the induction hypothesis, $G_1, \dots, G_t$ are all planar. We obtain a plane drawing of G by drawing $G_1, \dots, G_t$ in the plane side by side.

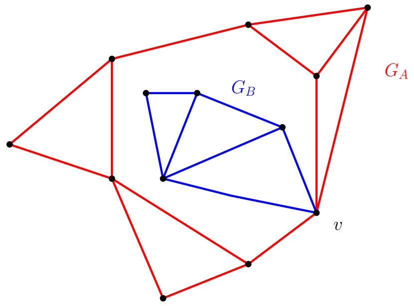
Next, suppose that G is connected, but not 2-connected. Then there exists a vertex $v \in V(G)$ such that $G \setminus v$ is disconnected. Let A be the vertex set of one component of $G \setminus v$, and let $B := V(G) \setminus (A \cup \{v\})$. Set $G_A := G[A \cup \{v\}]$ and $G_B := G[B \cup \{v\}]$. By the induction hypothesis, G_A and G_B are both planar. We draw G_A in the plane without any edge crossings, and we let f be some face of this drawing such that v lies on the

[3]We are using the fact that, by Lemma 2.3 from Lecture Notes 3, $K_5 \preceq_t G$ implies $K_5 \preceq_m G$.

[4]Here, $\{x, a, b\}$ and $\{y, x_i, x_j\}$ are the two sides of the bipartition of the subdivided $K_{3,3}$.

[5]We are using the fact that, by Lemma 2.3 from Lecture Notes 3, $K_{3,3} \preceq_t G$ implies $K_{3,3} \preceq_m G$.

boundary of f. We then draw G_B inside f, with v coinciding in the drawing of G_A and G_B.[6]



Suppose now that G is 2-connected, but not 3-connected. Since $|V(G)| \geq 4$, the fact that G is not 3-connected guarantees that there is a set $S \subseteq V(G)$ such that $|S| \leq 2$ and $G \setminus S$ is disconnected. Since G is 2-connected, we in fact have that $|S| = 2$; set $S = \{x, y\}$. Let A be the vertex set of some component of $G \setminus S$, and let $B := V(G) \setminus (A \cup S)$. Let $G_A := G[A \cup S] + xy$ and $G_B := G[B \cup S] + xy$.[7] Now, since G is 2-connected, each of $G[A \cup S]$ and $G[B \cup S]$ contains a path between x and y;[8] call these paths P_A and P_B, respectively. Clearly, $G_A \preceq_t G[A \cup S \cup V(P_B)]$ and $G_B \preceq_t G[B \cup S \cup V(P_A)]$; consequently, $G_A, G_B \preceq_t G$, and therefore (by Lemma 2.3 from Lecture Notes 3), $G_A, G_B \preceq_m G$. Since $K_5, K_{3,3} \not\preceq_m G$, Lemma 2.2 from Lecture Notes 3 guarantees that $K_5, K_{3,3} \not\preceq_m G_A$ and $K_5, K_{3,3} \not\preceq_m G_B$. By the induction hypothesis, G_A and G_B are both planar. We now draw G_A in the plane without edge crossings, and we let f be a face of this drawing such that the edge xy lies on the boundary of f. We now draw G_B inside f, with the edge xy coinciding in the drawing of G_A and G_B.[9] This way, we obtain a drawing of $G + xy$ in the plane without any edge crossings;[10] it follows that $G + xy$ is planar, and consequently, G is planar as well.

Finally, if G is 3-connected, then G is planar by Lemma 1.3. □

Lemma 1.4 proves the "(b) $\Longrightarrow$ (a)" part of Kuratowski's theorem. This completes our proof of Kuratowski's theorem.

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[6] This is slightly informal. The point is that we can stretch and shrink our drawing of G_B so that it "fits" inside of f.

[7] So, G_A is the graph with vertex set $A \cup S$ and edge set $E(G[A \cup S]) \cup \{xy\}$; if $xy \in E(G)$, then we simply have $G_A = G[A \cup S]$. Similar remarks apply to G_B.

[8] This follows from Proposition 1.1 of Lecture Notes 3. (Details?)

[9] Again, this is slightly informal. The point is that we can stretch and shrink our drawing of G_B so that it "fits" inside of f.

[10] As usual, $G + xy$ is the graph with vertex set $V(G)$ and edge set $E(G) \cup \{xy\}$. If $xy \in E(G)$, then we simply have that $G + xy = G$.

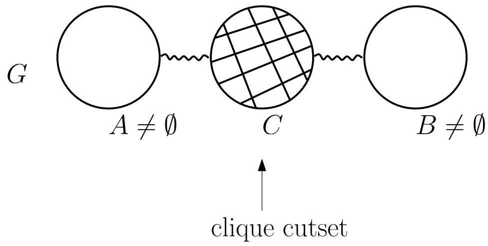
2 Hajós' Conjecture

In 1961, Hajós conjectured the following.

Hajós' Conjecture. *For every positive integer k, every graph of chromatic number at least k contains K_k as a topological minor.*

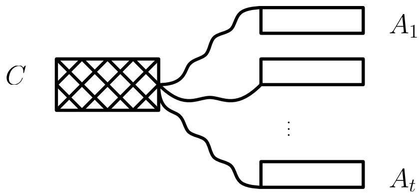
Hajós' Conjecture is obviously true for $k = 1$ and $k = 2$. For $k = 3$, we observe that if a graph G satisfies $\chi(G) \geq 3$, then G is not a forest, and in particular, G contains a cycle. Every cycle is a subdivision of K_3, i.e. every cycle contains K_3 as a topological minor. So, if $\chi(G) \geq 3$, then $K_3 \preceq_t G$. Hajós' Conjecture is also true for $k = 4$, as we now show.

A *clique-cutset* of a graph G is a clique $C \subsetneqq V(G)$ of G such that $G \setminus C$ is disconnected.[11] In particular, if G is disconnected, then $\emptyset$ is a clique-cutset of G.



Lemma 2.1. *Let G be a graph, and let C be a clique-cutset of G. Let $A_1, \dots, A_t$ be the vertex sets of the components of $G \setminus C$. Then $\chi(G) = \max\{\chi(G[A_1 \cup C]), \dots, \chi(G[A_t \cup C])\}$.*

Proof. To simplify notation, for all $i \in \{1, \dots, t\}$, set $G_i := G[A_i \cup C]$ and $\chi_i := \chi(G_i)$. We must show that $\chi(G) = \max\{\chi_1, \dots, \chi_t\}$. It is obvious that $\max\{\chi_1, \dots, \chi_t\} \leq \chi(G)$. It remains to show that $\chi(G) \leq \max\{\chi_1, \dots, \chi_t\}$.



For all $i \in \{1, \dots, t\}$, let $c_i : A_i \cup C \to \{1, \dots, \chi_i\}$ be a proper coloring of G_i. Since C is a clique of G, we know that for all $i \in \{1, \dots, t\}$, the coloring c_i assigns distinct colors to all vertices of C. So, after possibly permuting colors, we may assume that $c_1, \dots, c_t$ all agree on C. But now the union of $c_1, \dots, c_t$ is a proper coloring of G that uses at most $\max\{\chi_1, \dots, \chi_t\}$ colors, and we deduce that $\chi(G) \leq \max\{\chi_1, \dots, \chi_t\}$. □

[11] In some texts, a *clique-cutset* of G is defined to be a clique $C \subsetneqq V(G)$ of G such that $G \setminus C$ has more components than G. However, the definition that we gave above (requiring only that $G \setminus C$ be disconnected, regardless of the number of components of G) is more convenient for our purposes.

Theorem 2.2 (Dirac, 1952)**.** *Every graph of chromatic number at least* 4 *contains* K_4 *as a topological minor.*

Proof. Fix a graph G, and assume inductively that for all graphs G' with $|V(G')| < |V(G)|$, if $\chi(G') \geq 4$, then $K_4 \preceq_t G'$. We assume that $\chi(G) \geq 4$, and we show that $K_4 \preceq_t G$. We may assume that all proper induced subgraphs of G are 3-colorable,[12] for otherwise, the result follows from the induction hypothesis. In particular, this means that $\chi(G) = 4$.[13]

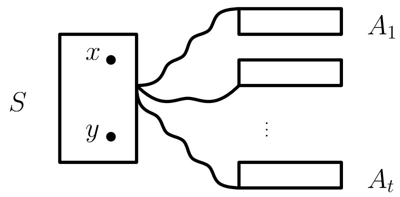
> **Claim 1.** G does not admit a clique-cutset. Furthermore, G is 2-connected.

Proof of Claim 1. The fact that G does not admit a clique-cutset readily follows from Lemma 2.1. Indeed, suppose C were a clique-cutset of G, and let $A_1, \dots, A_t$ be the vertex sets of $G \setminus C$. Then Lemma 2.1 guarantees that $\chi(G) = \max\{\chi(G[A_1 \cup C]), \dots, \chi(G[A_t \cup C])\}$. Since $\chi(G) = 4$, it follows that for some $i \in \{1, \dots, t\}$, we have that $\chi(G[A_i \cup C]) = 4$, contrary to the fact that all proper induced subgraphs of G are 3-colorable.

Clearly, $|V(G)| \geq \chi(G) = 4$. Furthermore, since G does not admit a clique-cutset, we see that G is connected and has no cut-vertices.[14] So, G is 2-connected. This proves Claim 1. ♦

> **Claim 2.** If G is not 3-connected, then $K_4 \preceq_t G$.

Proof of the Claim. Suppose that G is not 3-connected. Clearly, $|V(G)| \geq \chi(G) = 4$, and so (since G is not 3-connected) there exists a set $S \subseteq V(G)$ such that $|S| \leq 2$ and $G \setminus S$ is disconnected. By Claim 1, we have that $|S| = 2$ (say, $S = \{x, y\}$), and that the two vertices of S are non-adjacent.



Let $A_1, \dots, A_t$ $(t \geq 2)$ be the vertex sets of the components of $G \setminus S$, and for each $i \in \{1, \dots, t\}$, set $G_i := G[A_i \cup S]$. Then $\chi(G_i) \leq 3$ for all $i \in \{1, \dots, t\}$.[15]

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[12] A graph is *k-colorable* if it can be properly colored with at most k colors.

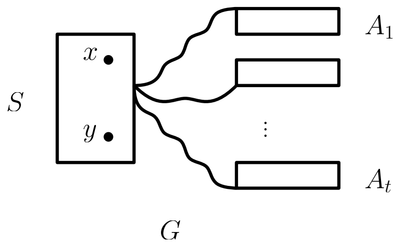
[13] Indeed, if $\chi(G) \geq 5$, then we fix any $v \in V(G)$, and we observe that $\chi(G \setminus v) \geq \chi(G) - 1 \geq 4$, contrary to the fact that all proper induced subgraphs of G are 3-colorable.

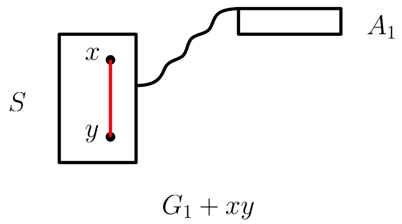
[14] A *cut-vertex* of a graph H is a vertex $v \in V(H)$ such that $H \setminus v$ is disconnected. Note that if v is a cut-vertex of H, then $\{v\}$ is a clique-cutset of H. So, a graph that does not admit a clique-cutset, has no cut-vertices.

[15] This is because all proper induced subgraphs of G are 3-colorable.

Suppose first that for all $i \in \{1, \dots, t\}$, there exists a 3-coloring c_i of G_i that assigns distinct colors to x and y.[16] After possibly permuting colors, we may assume that for all $i \in \{1, \dots, t\}$, we have that $c_i : A_i \cup S \to \{1, 2, 3\}$, $c_i(x) = 1$, and $c_i(y) = 2$. But now the union of $c_1, \dots, c_t$ is a proper coloring of G that uses at most three colors, contrary to the fact that $\chi(G) = 4$.

By symmetry, we may now assume that all 3-colorings of G_1 assign the same color to x and y. But then $\chi(G_1 + xy) = 4$.[17] So, by the induction hypothesis, we have that $K_4 \preceq_t G_1 + xy$.



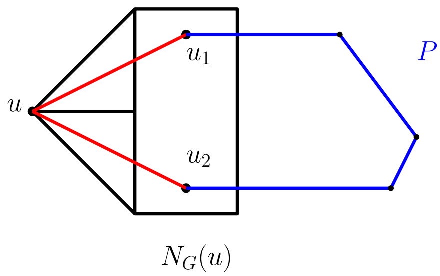


Now, since G is 2-connected, we see that each of x, y has a neighbor in A_2,[18] and so there exists an induced path P in G_2 between x and y. But now $G[A_1 \cup V(P)]$ is a subdivision of $G_1 + xy$, and so $G_1 + xy \preceq_t G$. Since $K_4 \preceq_t G_1 + xy$, we have that $K_4 \preceq_t G$. This proves Claim 2. ♦

In view of Claim 2, we may now assume that G is 3-connected.

> **Claim 3.** Either G contains a cycle of length at least four, or $K_4 \preceq_t G$.

Proof of Claim 3. Since G is 3-connected, we have that $\delta(G) \geq 3$. Now, fix any vertex u of G; then $d_G(u) \geq \delta(G) \geq 3$. If $N_G(u)$ is a clique, then G contains a K_4 as a subgraph,[19] and therefore as a topological minor, and we are done. So, we may assume that some two neighbors (call them u_1 and u_2) of u are non-adjacent.



[16] A *k-coloring* of a graph is a proper coloring of that graph that uses at most k colors.

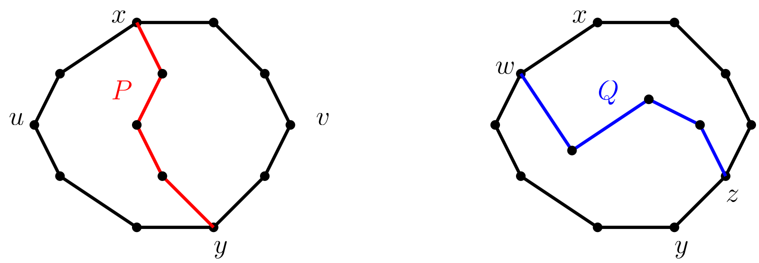
[17] Indeed, since $\chi(G_1) \leq 3$, it is obvious that $\chi(G_1 + xy) \leq 4$. If $\chi(G_1 + xy) \leq 3$, then we fix some 3-coloring of $G_1 + xy$, and we observe that this coloring must assign different colors to x and y (because x and y are adjacent in $G_1 + xy$). But now this coloring is a 3-coloring of G that assigns distinct colors to x and y, a contradiction.

[18] This follows from Proposition 1.1 from Lecture Notes 3.

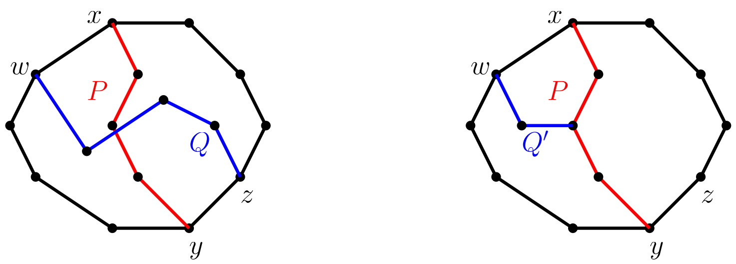
[19] Indeed, take u and any three of its neighbors.

Since G is 3-connected, we know that $G \setminus u$ is connected, and consequently, $G \setminus u$ contains a path P between u_1 and u_2. But now $u - u_1 - P - u_2 - u$ is a cycle of length at least four in G.[20] This proves Claim 3. ⧫

In view of Claim 3, we may assume that G contains a cycle C of length at least four. Let u and v be some non-consecutive vertices of C. Since G is 3-connected, we know that $G \setminus \{u, v\}$ is connected; let P be a shortest path in $G \setminus \{u, v\}$ between the two components of $C \setminus \{u, v\}$, and let x and y be the two endpoints of P. (Note that $x, y \in V(C)$, and no internal vertex of P belongs to C. Furthermore, note that x and y are not consecutive vertices of the cycle C.) Since G is 3-connected, $G \setminus \{x, y\}$ is connected; let Q be a shortest path in $G \setminus \{x, y\}$ between the two components of $C \setminus \{x, y\}$, and let w and z be the two endpoints of Q.



Now, if P and Q do not intersect, then $C \cup P \cup Q$ is a subdivision of K_4,[21] and so $K_4 \preceq_t G$. It remains to consider the case when P and Q do intersect. Let Q' be the subpath of Q from w to the first intersection of P and Q. But now $C \cup P \cup Q'$ is a subdivision of K_4, and so $K_4 \preceq_t G$.



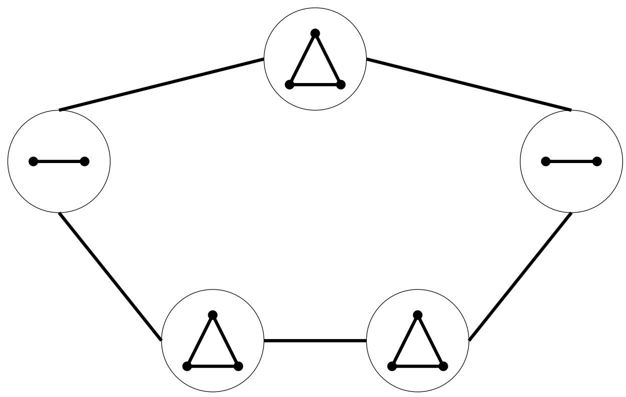
□

In 1979 Catlin proved that Hajós' Conjecture fails for $k \geq 7$, as the example below shows.[22]

[20] We are using the fact that $u_1 u_2 \notin E(G)$, and so P has at least one internal vertex.

[21] Here, $C \cup P \cup Q$ is the graph whose vertex set is $V(C) \cup V(P) \cup V(Q)$, and whose edge set is $E(C) \cup E(P) \cup E(Q)$.

[22] A line between two circles indicates that all vertices inside one of the circles are adjacent to all vertices inside the other circle.



Indeed, the graph above has chromatic number 7, and yet it does not contain K_7 as a topological minor.[23] For $k \geq 8$, we can obtain a counterexample to Hajós' Conjecture by adding $k-7$ universal vertices (i.e. vertices adjacent to all other vertices of the graph) to the graph above. Hajós' Conjecture is open for $k = 5$ and $k = 6$.

3 Hadwiger's Conjecture

In 1943, Hadwiger conjectured the following.

Hadwiger's Conjecture. *For every positive integer k, every graph of chromatic number at least k contains K_k as a minor.*

Since a topological minor is a special case of a minor (by Lemma 2.3 from Lecture Notes 3), Hadwiger's Conjecture is weaker than Hajós' Conjecture. Thus, since Hajós' Conjecture is true for $k \leq 4$, Hadwiger's conjecture is also true for $k \leq 4$. Hadwiger's Conjecture for $k = 5$ is equivalent to the famous Four Color Theorem (proven by Appel and Haken in 1976), which states that every planar graph is 4-colorable.[24] Further, in 1993, Robertson, Seymour, and Thomas proved that Hadwiger's Conjecture is true for $k = 6$. For $k \geq 7$, the conjecture remains open.

[23] Check this!

[24] The equivalence of Hadwiger's Conjecture for $k = 5$ and the Four Color Theorem is not entirely obvious, though, and we omit the details.