NDMI012: Combinatorics and Graph Theory 2

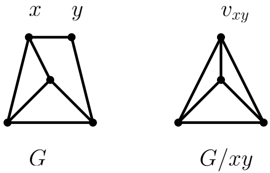
Lecture #3

Minors and planar graphs (part I)

Irena Penev

March 1, 2022

- Given a graph G and an edge $xy \in E(G)$, we denote by G/xy the graph obtained from G by contracting xy to a vertex v_{xy}.



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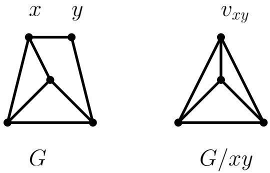

- For a non-negative integer k, a graph G is *k-connected* if it satisfies the following two conditions:
  - $|V(G)| \geq k + 1$;
  - for all $S \subseteq V(G)$ such that $|S| \leq k - 1$, the graph $G \setminus S$ is connected.

Proposition 1.1

Let k be a positive integer, let G be a k-connected graph, and let $S \subseteq V(G)$ be such that $|S| = k$. Then every vertex of S has a neighbor in each component of $G \setminus S$.

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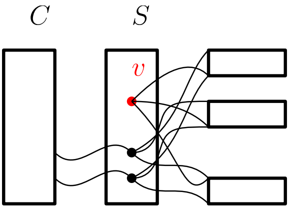

Details: Lecture Notes.

Lemma 1.2

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> ***Claim.*** *For all $xy \in E(G)$, either G/xy is 3-connected, or there exists a vertex $z \in V(G) \setminus \{x, y\}$ such that $G \setminus \{x, y, z\}$ is disconnected.*

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Claim. *For all $xy \in E(G)$, either G/xy is 3-connected, or there exists a vertex $z \in V(G) \setminus \{x, y\}$ such that $G \setminus \{x, y, z\}$ is disconnected.*

Proof of the Claim (outline). Fix $xy \in E(G)$, and suppose that G/xy is not 3-connected. Clearly, G/xy has at least four vertices, and if S is a cutset of G/xy of size at most two, then it must contain v_{xy}, and then $(S \setminus \{v_{xy}\}) \cup \{x, y\}$ is the cutset that we need. This proves the Claim.

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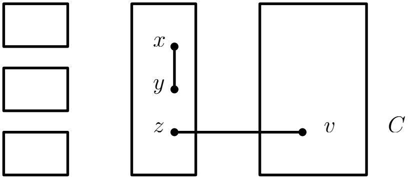
Proof (continued). Since G is 3-connected, it is clear that G has at least one edge. Now, suppose that for all $e \in E(G)$, the graph G/e is not 3-connected. Then using the Claim, we fix an edge $xy \in E(G)$ and a vertex $z \in V(G) \setminus \{x, y\}$ such that $G \setminus \{x, y, z\}$ is disconnected, and we fix a component C of $G \setminus \{x, y, z\}$; we may assume that xy, z, C were chosen so that $|V(C)|$ is minimum.



Using Proposition 1.1, we let $v \in V(C)$ be a neighbor of z.

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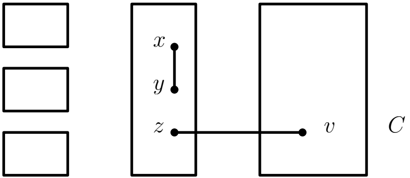

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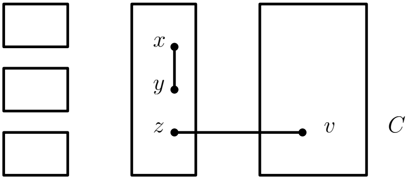

By our supposition, G/zv is not 3-connected, and so by the Claim, there exists some $w \in V(G) \setminus \{z, v\}$ such that $G \setminus \{z, v, w\}$ is disconnected.

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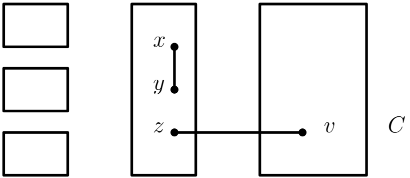

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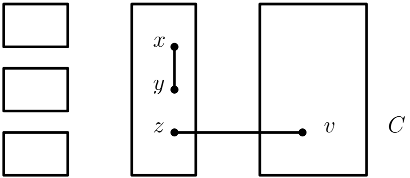

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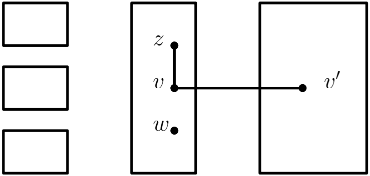

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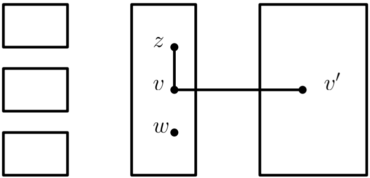

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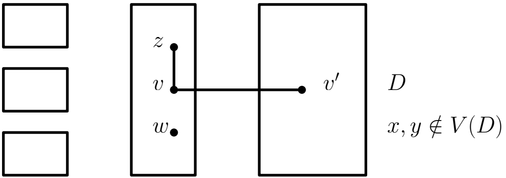

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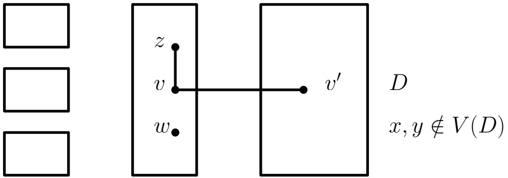

Now, let us show that $V(D) \subsetneq V(C)$. By Proposition 1.1 we know that v has a neighbor v' in $V(D)$. But note that all neighbors of v in G belong to $V(C) \cup \{x, y, z\}$, and so since $x, y, z \notin V(D)$, we have that $v' \in V(D) \cap V(C)$.

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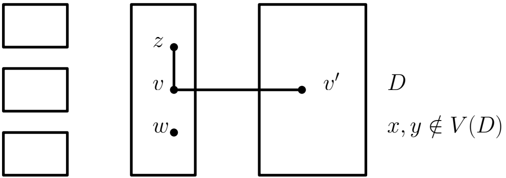

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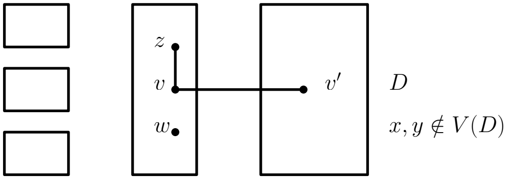
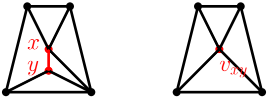


Now, let us show that $V(D) \subsetneqq V(C)$. By Proposition 1.1 we know that v has a neighbor v' in $V(D)$. But note that all neighbors of v in G belong to $V(C) \cup \{x, y, z\}$, and so since $x, y, z \notin V(D)$, we have that $v' \in V(D) \cap V(C)$. Since C is a component and D a connected induced subgraph of $G \setminus \{x, y, z\}$, we now deduce that $V(D) \subseteq V(C)$. Since $v \in V(C) \setminus V(D)$, it follows that $V(D) \subsetneqq V(C)$. But this contradicts the minimality of C.

Proposition 1.3

Let G be a graph, and let $xy \in E(G)$ be such that $d_G(x), d_G(y) \geq 3$. If G/xy is 3-connected, then so is G.



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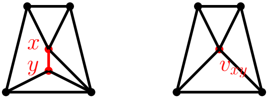

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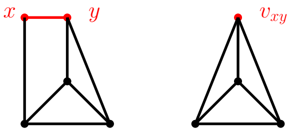
- The $d_G(x), d_G(y) \geq 3$ condition is necessary because every 3-connected graph G satisfies $\delta(G) \geq 3$.



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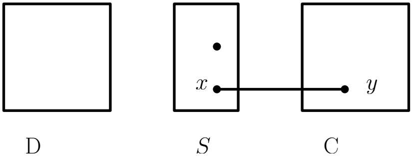

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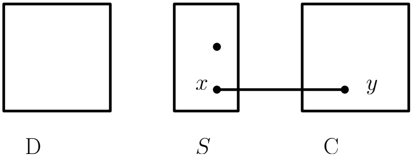

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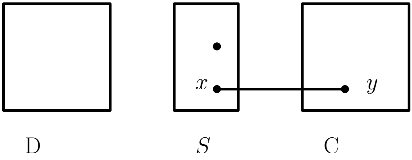

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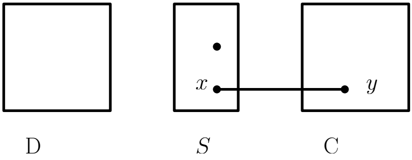

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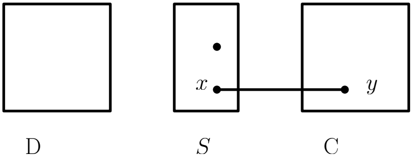

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Theorem 1.4 [Tutte, 1961]

A graph G is 3-connected if and only if there exists a sequence $G_0, \ldots, G_n$ of graphs with the following properties:

(1) $G_0 \cong K_4$ and $G = G_n$;

(2) for all $i \in \{0, \ldots, n-1\}$, G_{i+1} has an edge xy with $d_{G_{i+1}}(x), d_{G_{i+1}}(y) \geq 3$ and $G_i = G_{i+1}/xy$.

Proof. This follows from Lemma 1.2 and Proposition 1.3 (details: Lecture Notes).

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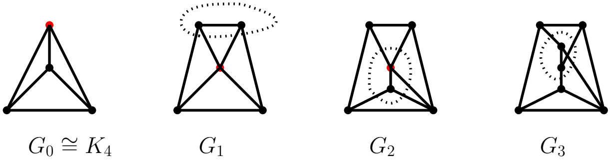
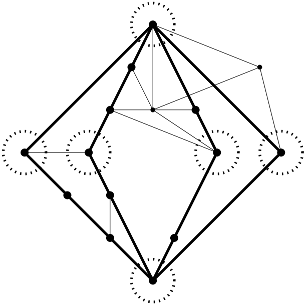
- Theorem 1.4 guarantees that every 3-connected graph can be obtained from K_4 by repeatedly "decontracting" vertices into edges, making sure that, at each step, both new vertices have degree at least three.



Definition

A graph H is a *topological minor* of a graph G, and we write $H \preceq_t G$, if G contains some subdivision of H as a subgraph. The vertices of this subdivision that correspond to the vertices of H are called *branch vertices*.

- The graph below contains $K_{2,4}$ as a topological minor.



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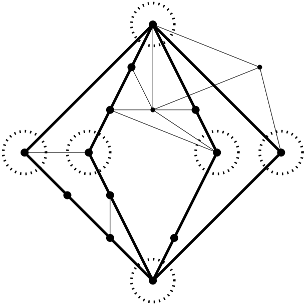


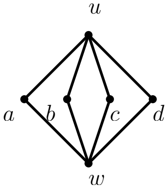
- The topological minor relation is transitive, that is, for all graphs G_1, G_2, G_3, if $G_1 \preceq_t G_2$ and $G_2 \preceq_t G_3$, then $G_1 \preceq_t G_3$.

Definition

A graph H is a *minor* of a graph G, and we write $H \preceq_m G$, if there exists a family $\{X_v\}_{v \in V(H)}$ of pairwise disjoint, non-empty subsets of $V(G)$, called *branch sets*, such that

- $G[X_v]$ is connected for all $v \in V(H)$, and
- for all $uv \in E(H)$, there is an edge between X_u and X_v in G.

- For example, the graph below (on the right) contains $K_{2,4}$ as a minor.



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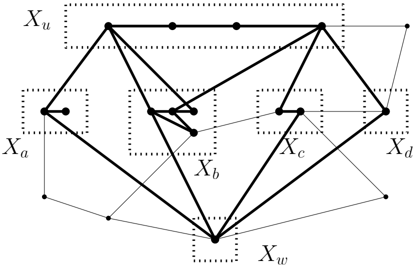

- Our goal is to prove the following theorem, called "Kuratowski's theorem," or sometimes the "Kuratowski-Wagner theorem."

Theorem 3.3 [Kuratowski, 1930; Wagner, 1937]

Let G be a graph. Then the following are equivalent:

(a) G is planar;

(b) G contains neither K_5 nor $K_{3,3}$ as a minor;

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- We will prove some preliminary results that we need for this theorem today. We will complete the proof next time.

Lemma 2.1

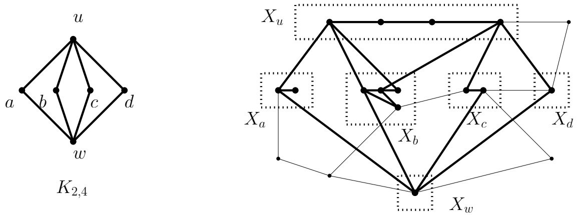
For all graphs G and H, the following are equivalent:

(1) $H \preceq_m G$;

(2) G can be transformed into (an isomorphic copy of) H by a sequence of vertex deletions, edge deletions, and edge contractions;

(3) there exists a subgraph G' of G such that G' can be transformed into (an isomorphic copy) of H by a sequence of edge contractions.

Proof. Lecture Notes.



Lemma 2.2

The minor relation is transitive, that is, for all graphs G_1, G_2, G_3, if $G_1 \preceq_m G_2$ and $G_2 \preceq_m G_3$, then $G_1 \preceq_m G_3$.

Proof.

Lemma 2.2

The minor relation is transitive, that is, for all graphs G_1, G_2, G_3, if $G_1 \preceq_m G_2$ and $G_2 \preceq_m G_3$, then $G_1 \preceq_m G_3$.

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- Lemma 2.2 can also be proven directly, using the definition of a minor.
  - Proof?

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For all graphs G and H, if $H \preceq_t G$, then $H \preceq_m G$.

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Proof. Fix graphs G and H, and assume that $H \preceq_t G$. Then G contains a subgraph G' that is isomorphic to a subdivision of H, and clearly, H can be obtained from the subgraph G' by a sequence of edge contractions. Now Lemma 2.1 guarantees that $H \preceq_m G$.

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- Note that the converse of Lemma 2.3 is false, i.e. it is possible that $H \preceq_m G$, but $H \not\preceq_t G$.

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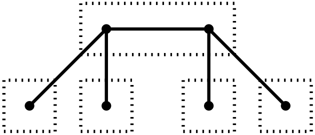
- Note that the converse of Lemma 2.3 is false, i.e. it is possible that $H \preceq_m G$, but $H \not\preceq_t G$.
- For example, the graph below contains $K_{1,4}$ as a minor, but not as a topological minor.



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Let G and H be graphs such that $H \preceq_m G$ and $\Delta(H) \leq 3$. Then $H \preceq_t G$.

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- if $uv \in E(H)$, then there is exactly one edge between X_u and X_v in G',
- if $uv \notin E(H)$, then there are no edges between X_u and X_v.

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- if $uv \in E(H)$, then there is exactly one edge between X_u and X_v in G',
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By the minimality of G', $G'[X_v]$ is a tree.

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Proof (continued).

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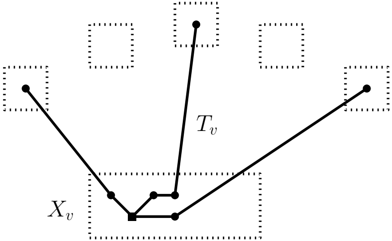
Proof (continued). Now, for each $v \in V(H)$, we let T_v be the graph obtained from $G'[X_v]$ by adding to it the edges between X_v and $V(G') \setminus X_v$ (and the endpoints of those edges).



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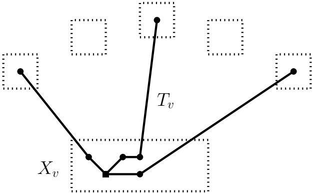

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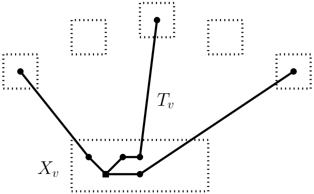

Clearly, for each $v \in V(H)$, the graph T_v is a tree. Since $\Delta(H) \leq 3$, the minimality of G' guarantees that T_v has at most three leaves, and so $\Delta(T_v) \leq 3$. Moreover, T_v has at most one vertex of degree three, and if this vertex exists, then it belongs to X_v.

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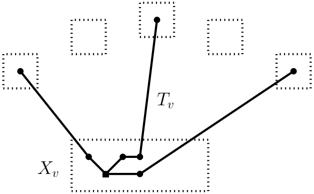

Now, for all $v \in V(H)$, we let v' be the unique vertex of T_v of degree three if such a vertex exists, and otherwise, we let v' be any vertex in X_v.

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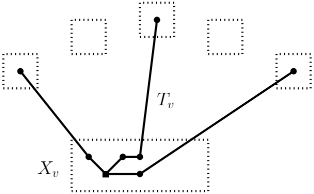

Now, for all $v \in V(H)$, we let v' be the unique vertex of T_v of degree three if such a vertex exists, and otherwise, we let v' be any vertex in X_v. It is now clear that G' is a subdivision of H (vertices v' are the branch vertices), and so $H \preceq_t G$.

Lemma 2.5

Let G be a graph. Then the following are equivalent:

(1) G contains at least one $K_5, K_{3,3}$ as a topological minor;

(2) G contains at least one $K_5, K_{3,3}$ as a minor.

Proof (outline).

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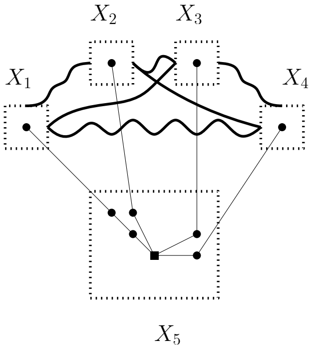
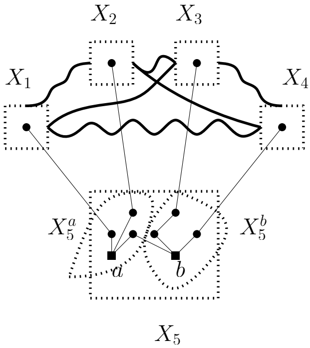


or



Definition

A graph is *planar* if it can be drawn in the plane without any edge crossings.

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- Obviously, a graph can be drawn in the plane without any edge crossings if and only if it can be drawn on a sphere without any edge crossings.
- So, planar graphs are precisely those that can be drawn on a sphere without any edge crossings.

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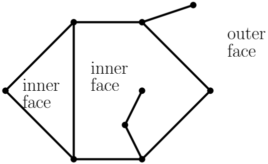
- When we draw a graph on a plane without edge crossings, we divide the plane into regions, called *faces*; one of the faces, called the *outer face* is unbounded, and the remaining faces (called *inner faces*) are bounded.



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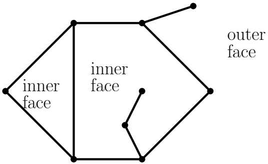

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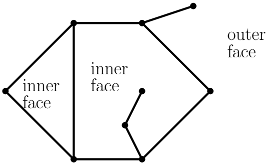

- We can define faces on a sphere analogously, but in this case, all faces are bounded, and we get no asymmetry between the inner faces and the outer face.
- For this reason, for proving theorems, it is often more practical to draw on a sphere than on a plane.

Lemma 3.1

If a graph is planar, then so are all its minors.

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Proof. Clearly, any graph obtained from a planar graph by deleting one vertex, deleting one edge, or contracting one edge is planar. So, by Lemma 2.1, all minors of a planar graph are planar.

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A *homeomorphism* of the sphere is a bijection f from the sphere to itself such that both f and f^{-1} are continuous.

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- Informally, a homeomorphism of the sphere is the result of "stretching" the sphere (and possibly also rotating and taking mirror images).
- Two graph drawings on the sphere are *equivalent* if some sphere homeomorphism transforms one drawing into the other.

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Graphs K_5 and $K_{3,3}$ are not planar. Consequently, no planar graph contains K_5 or $K_{3,3}$ as a minor.

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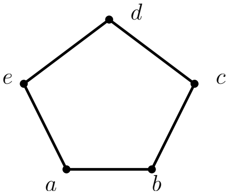
Proof. Suppose that K_5 is planar, so that we can draw it on a sphere without any edge crossings. Let $\{a, b, c, d, e\}$ be the vertex set of the K_5. We first draw the 5-cycle a, b, c, d, e, a on the sphere.



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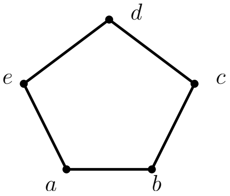

Since edges ac and bd do not cross, we must draw them through distinct faces created by our 5-cycle a, b, c, d, e, a, and we obtain the following (next slide).

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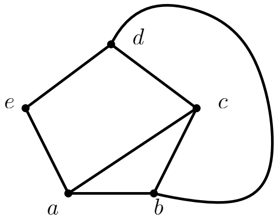

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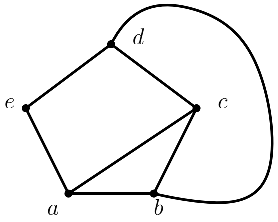

There is now only one way to add the edge ce to our drawing without creating edge crossings (next slide).

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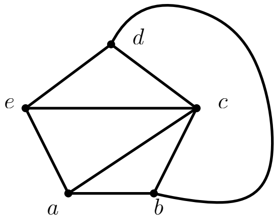

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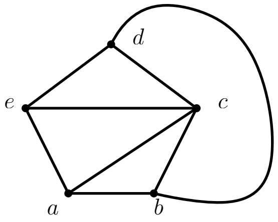

Further, there is only one way to add the edge *ad* to our drawing without creating edge crossings (next slide).

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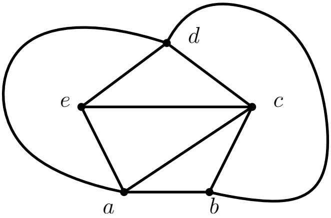

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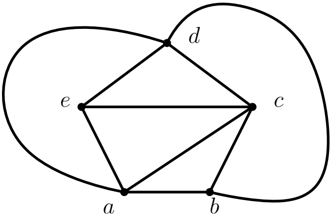

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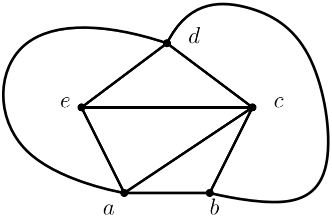

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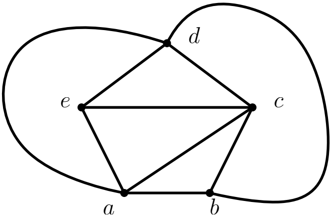

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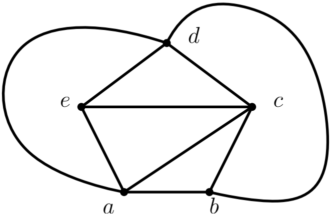

But now it is not possible to add the edge *be* to our drawing without creating edge crossings.

So, K_5 is not planar. A similar argument shows that $K_{3,3}$ is not planar.

Since K_5 and $K_{3,3}$ are not planar, Lemma 3.1 guarantees that no planar graph contains K_5 or $K_{3,3}$ as a minor.

Theorem 3.3 [Kuratowski, 1930; Wagner, 1937]

Let G be a graph. Then the following are equivalent:

(a) G is planar;

(b) G contains neither K_5 nor $K_{3,3}$ as a minor;

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- We will do this in the next lecture.