NDMI012: Combinatorics and Graph Theory 2

Lecture #3
Minors and planar graphs (part I)

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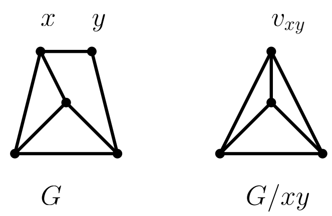
1 3-connected graphs

Given a graph G and an edge $xy \in E(G)$, we denote by G/xy the graph obtained from G by contracting xy to a vertex v_{xy}.



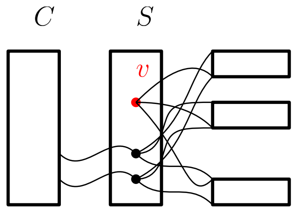
More formally, G/xy is the graph with vertex set $V(G/xy) = (V(G) \setminus \{x,y\}) \cup \{v_{xy}\}$ (where $v_{xy} \notin V(G)$) and edge set $E(G) = \{e \in \binom{V(G) \setminus \{x,y\}}{2} \mid e \in E(G)\} \cup \{vv_{xy} \mid v \in V(G) \setminus \{x,y\}$, and either $vx \in E(G)$ or $vy \in E(G)\}$. If $e = xy$, then we sometimes write G/e instead of G/xy, and v_e instead of v_{xy}.

Recall that for a non-negative integer k, a graph G is *k-connected* if it satisfies the following two conditions:

- $|V(G)| \geq k+1$;
- for all $S \subseteq V(G)$ such that $|S| \leq k-1$, the graph $G \setminus S$ is connected.

Proposition 1.1. *Let k be a positive integer, let G be a k-connected graph, and let $S \subseteq V(G)$ be such that $|S| = k$. Then every vertex of S has a neighbor in each component of $G \setminus S$.*

Proof. Suppose otherwise, and fix a vertex $v \in S$ and a component C of $G \setminus S$ such that v has no neighbors in $V(C)$. Then $S \setminus \{v\}$ separates v from $V(C)$ in G, and in particular, $G \setminus (S \setminus \{v\})$ is disconnected. But this is impossible since $|S \setminus \{v\}| = k-1$ and G is k-connected.



□

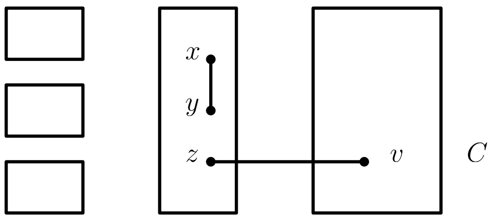
Lemma 1.2. *Let G be a 3-connected graph on more than four vertices. Then G has an edge e such that G/e is 3-connected.*

Proof.

> **Claim.** For all $xy \in E(G)$, either G/xy is 3-connected, or there exists a vertex $z \in V(G) \setminus \{x, y\}$ such that $G \setminus \{x, y, z\}$ is disconnected.

Proof of the Claim. Fix $xy \in E(G)$, and suppose that G/xy is not 3-connected. Clearly, G/xy has at least four vertices,[1] and so there exists some $S \subseteq V(G/xy)$ such that $|S| \leq 2$ and $(G/xy) \setminus S$ is disconnected. If $v_{xy} \notin S$, then it is clear that $G \setminus S$ is disconnected, contrary to the fact that G is 3-connected. So, $v_{xy} \in S$. Now set $S' = (S \setminus \{v_{xy}\}) \cup \{x, y\}$. Then $|S'| = |S| + 1$ and $G \setminus S' = (G/xy) \setminus S$; so, $G \setminus S'$ is disconnected. Since G is 3-connected, it follows that $|S'| \geq 3$; since $|S| \leq 2$, we deduce that $|S'| = 3$, and the result follows.[2] This proves the Claim. ♦

Since G is 3-connected, it is clear that G has at least one edge. Now, suppose that for all $e \in E(G)$, the graph G/e is not 3-connected. Then using the Claim, we fix an edge $xy \in E(G)$ and a vertex $z \in V(G) \setminus \{x, y\}$ such that $G \setminus \{x, y, z\}$ is disconnected, and we fix a component C of $G \setminus \{x, y, z\}$; we may assume that xy, z, C were chosen so that $|V(C)|$ is minimum.[3]



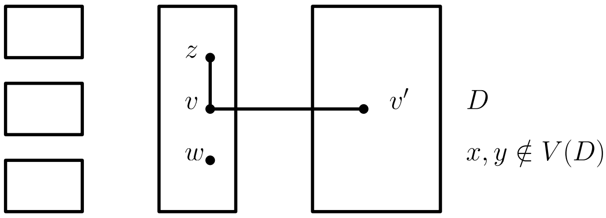
Using Proposition 1.1, we let $v \in V(C)$ be a neighbor of z. By our supposition, G/zv is not 3-connected, and so by the Claim, there exists some $w \in$

[1] This is because $|V(G)| > 4$, and clearly, $|V(G/xy)| = |V(G)| - 1$.

[2] Indeed, we take z to be the (unique) vertex of $S' \setminus \{x, y\}$.

[3] So, we are assuming that for all edges $x'y' \in E(G)$, all vertices $z' \in V(G) \setminus \{x', y'\}$ such that $\{x', y', z'\}$ is disconnected, and all components C' of $G \setminus \{x', y', z'\}$, we have that $|V(C)| \leq |V(C')|$.

$V(G) \setminus \{z, v\}$ such that $G \setminus \{z, v, w\}$ is disconnected.[4] Since $xy \in E(G)$, there exists a component D of $G \setminus \{z, v, w\}$ such that $x, y \notin V(D)$; so, D is in fact a component of $G \setminus \{x, y, z, v, w\}$, and in particular, it is a connected induced subgraph of $G \setminus \{x, y, z\}$.



Now, let us show that $V(D) \subsetneq V(C)$. By Proposition 1.1,[5] we know that v has a neighbor v' in $V(D)$. But note that all neighbors of v in G belong to $V(C) \cup \{x, y, z\}$, and so since $x, y, z \notin V(D)$,[6] we have that $v' \in V(D) \cap V(C)$. Since C is a component and D a connected induced subgraph of $G \setminus \{x, y, z\}$, we now deduce that $V(D) \subseteq V(C)$. Since $v \in V(C) \setminus V(D)$, it follows that $V(D) \subsetneq V(C)$. But this contradicts the minimality of C. □

Proposition 1.3. *Let G be a graph, and let $xy \in E(G)$ be such that $d_G(x), d_G(y) \geq 3$. If G/xy is 3-connected, then so is G.*

Proof. To simplify notation, set $G' := G/xy$. Assume that G' is 3-connected. Then by definition, G' has at least four vertices, and consequently, G has at least five vertices.

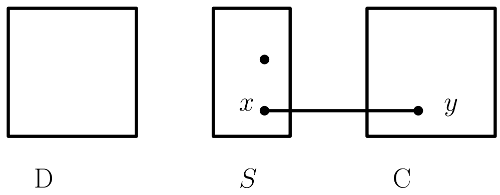
Now, fix $S \subseteq V(G)$ such that $|S| \leq 2$; we must show that $G \setminus S$ is connected. If $S \cap \{x, y\} = \emptyset$, then $(G \setminus S)/xy = G' \setminus S$; since G' is 3-connected, we see that $G' \setminus S$ is connected, and we deduce that $(G \setminus S)/xy$ is connected. But then clearly, $G \setminus S$ is also connected. Next, if $S = \{x, y\}$, then $G \setminus S = G' \setminus v_{xy}$; since G' is 3-connected, we know that $G' \setminus v_{xy}$ is connected, and so $G \setminus S$ is connected.

It remains to consider the case when S contains exactly one of x, y. By symmetry, we may assume that $x \in S$ and $y \notin S$. Now, suppose that $G \setminus S$ is disconnected. Let C be the component of $G \setminus S$ that contains y, and let D be some other component of $G \setminus S$. Clearly, $N_G(y) \subseteq S \cup (V(C) \setminus \{y\})$; since $d_G(y) \geq 3$ and $|S| \leq 2$, we see that $V(C) \setminus \{y\} \neq \emptyset$. Set $S' := (S \setminus \{x\}) \cup \{v_{xy}\}$, and note that $G \setminus (S \cup \{y\}) = G' \setminus S'$. But now S' separates $V(C) \setminus \{y\} \neq \emptyset$ from $V(D)$ in G', contrary to the fact that G' is 3-connected and $|S'| \leq 2$.

[4] It is possible that $w \in \{x, y\}$.

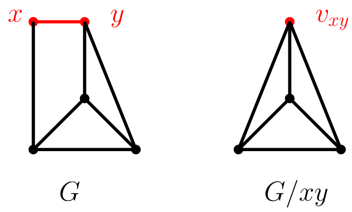
[5] We are applying Proposition 1.1 to G, $k = 3$, and $S = \{z, v, x\}$.

[6] We already saw that $x, y \notin V(D)$. Since D is a component of $G \setminus \{z, v, w\}$, we also have that $z \notin V(D)$. So, $x, y, z \notin V(D)$.



□

Note that in the statement of Proposition 1.3, the requirement that $d_G(x), d_G(y) \geq 3$ is necessary, since every 3-connected graph G satisfies $\delta(G) \geq 3$.[7] For a concrete example, see the picture below (G/xy is 3-connected, but G is not).



Theorem 1.4 (Tutte, 1961)**.** *A graph G is 3-connected if and only if there exists a sequence $G_0, \dots, G_n$ of graphs with the following properties:*

(1) $G_0 \cong K_4$ and $G = G_n$;

(2) for all $i \in \{0, \dots, n-1\}$, G_{i+1} has an edge xy with $d_{G_{i+1}}(x), d_{G_{i+1}}(y) \geq 3$ and $G_i = G_{i+1}/xy$.

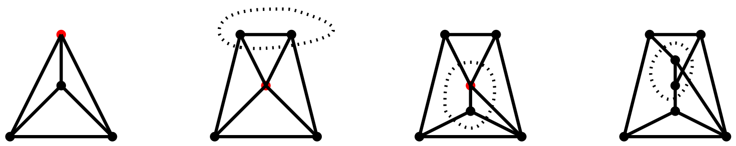
Proof. Fix a graph G.

By definition, all 3-connected graphs have at least four vertices, and it is easy to see that K_4 is (up to isomorphism) the only 3-connected graph on four vertices. So, if G is 3-connected, then Lemma 1.2 and an easy induction guarantee that there exists a sequence $G_0, \dots, G_n$, as in the statement of the theorem.

On the other hand, if there exists a sequence $G_0, \dots, G_n$ as in the statement of the theorem, then Proposition 1.3 and an easy induction guarantee that G is 3-connected. □

Note that Theorem 1.4 guarantees that every 3-connected graph can be obtained from K_4 by repeatedly “decontracting” vertices into edges, making sure that, at each step, both new vertices have degree at least three. An example is shown below (at each step, the vertex to be “decontracted” is in red, and in the subsequent step, the edge obtained by this “decontraction” is in a dotted bag); each graph in the sequence is 3-connected.

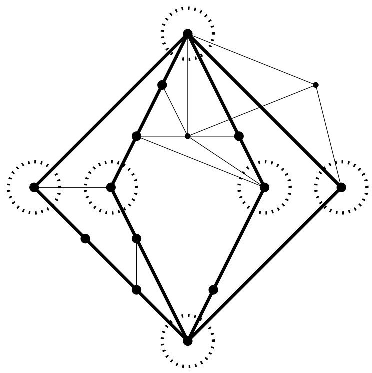
[7] Otherwise, we take a vertex $v \in V(G)$ with $d_G(v) \leq 2$, and we observe that $N_G(v)$ separates v from $V(G) \setminus N_G[v]$ (this is non-empty because $|N_G[v]| \leq 3$, and 3-connected graphs have at least four vertices), contrary to the fact that $|N_G(v)| = d_G(v) \leq 2$ and G is 3-connected.



$G_0 \cong K_4$ G_1 G_2 G_3

2 Minors and topological minors

A graph H is a *topological minor* of a graph G, and we write $H \preceq_t G$, if G contains some subdivision of H as a subgraph.[8] The vertices of this subdivision that correspond to the vertices of H are called *branch vertices*.[9] For example, the graph below contains $K_{2,4}$ as a topological minor (the branch vertices are in dotted circles).



Obviously, the topological minor relation is transitive, that is, for all graphs G_1, G_2, G_3, if $G_1 \preceq_t G_2$ and $G_2 \preceq_t G_3$, then $G_1 \preceq_t G_3$.

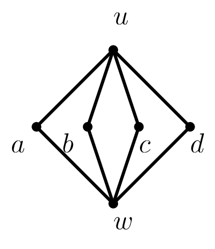
A graph H is a *minor* of a graph G, and we write $H \preceq_m G$, if there exists a family $\{X_v\}_{v \in V(H)}$ of pairwise disjoint, non-empty subsets of $V(G)$, called *branch sets*, such that

- $G[X_v]$ is connected for all $v \in V(H)$, and
- for all $uv \in E(H)$, there is an edge between X_u and X_v in G.

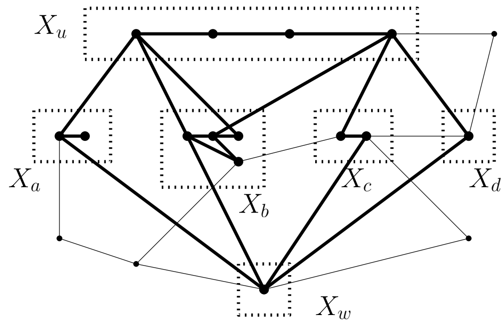
For example, the graph below (on the right) contains $K_{2,4}$ as a minor.

[8] Every graph is considered to be a subdivision of itself.

[9] If $\delta(H) \geq 3$, then branch vertices are uniquely defined. Otherwise, they need not be uniquely defined.



$K_{2,4}$



Lemma 2.1. *For all graphs G and H, the following are equivalent:*

(1) $H \preceq_m G$;

(2) G can be transformed into (an isomorphic copy of) H by a sequence of vertex deletions, edge deletions, and edge contractions;[10]

(3) there exists a subgraph G' of G such that G' can be transformed into (an isomorphic copy) of H by a sequence of edge contractions.[11]

Proof. Fix graphs G and H.

Suppose first that (1) holds, and let $\{X_v\}_{v \in V(H)}$ be the family of branch sets of the H minor in G. Let G' be the subgraph of G obtained by first deleting $V(G) \setminus \left(\bigcup_{v \in V(H)} X_v \right)$, and then for all distinct $u, v \in V(H)$ such that $uv \notin E(H)$, deleting all the edges between X_u and X_v. Let G'' be the graph obtained from G' by contracting each X_v into a vertex (we contract the X_v's one edge at a time, in any order). Clearly, $G'' \cong H$. So, (3) holds.

Suppose now that (3) holds, and let G' be a subgraph of G such that G' can be transformed into (an isomorphic copy) of H by a sequence of edge contractions. Let $G_0, \dots, G_\ell$ be a sequence of graphs such that $G_0 = G'$, $G_\ell \cong H$, and for all $i \in \{0, \dots, \ell - 1\}$, G_{i+1} can be obtained from G_i by contracting one edge. We may assume that $G_\ell = H$ (we rename vertices if necessary). For all $v \in V(H)$, we set $X_v^\ell = \{v\}$. Next, for all $i \in \{0, \dots, \ell-1\}$, having defined the sets X_v^{i+1}, we define the sets X_v^i as follows. Let $u_1u_2 \in E(G_i)$ be the edge of G_i that was contracted to obtain G_{i+1}, and let u be the vertex formed by contracting that edge.[12] For all $v \in V(H)$, if $u \in X_v^{i+1}$, then we set $X_v^i := (X_v^{i+1} \setminus \{u\}) \cup \{u_1, u_2\}$, and otherwise, we set $X_v^i := X_v^{i+1}$. It then follows by an easy induction that for all $i \in \{0, \dots, \ell\}$, $\{X_v^i\}_{v \in V(H)}$ is a family of branch sets for the H minor in G_i. In particular, $\{X_v^0\}_{v \in V(H)}$ is a family of branch sets for the H minor in $G_0 = G'$, and therefore (since G' is a subgraph of G) in G as well. So, (1) holds.

It is clear that (3) implies (2). Further, it is clear that if a graph G_2 is obtained from a graph G_1 by first contracting an edge and then deleting a

[10] Possibly, $G \cong H$.

[11] Possibly, $G' = G$ or $G' \cong H$.

[12] So, $u = v_{u_1u_2}$.

vertex or an edge, then we can also obtain G_2 from G_1 by first deleting one or more vertices or edges, and then possibly contracting an edge.[13] Thus, if H can be obtained from G by a sequence of vertex deletions, edge deletions, and edge contractions, then H can be obtained from G by first (possibly) deleting some vertices or edges (thus obtaining a subgraph G' of G), and then (possibly) contracting edges of G'. So, (2) implies (3).

We have now proven that (1), (2), and (3) are equivalent. $\square$

Lemma 2.2. *The minor relation is transitive, that is, for all graphs G_1, G_2, G_3, if $G_1 \preceq_m G_2$ and $G_2 \preceq_m G_3$, then $G_1 \preceq_m G_3$.*

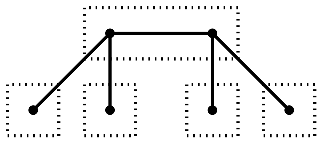
Proof. Fix graphs G_1, G_2, G_3 such that $G_1 \preceq_m G_2$ and $G_2 \preceq_m G_3$. By Lemma 2.1, G_1 can be obtained from G_2 by a sequence of vertex deletions, edge deletions, and edge contractions, and G_2 can similarly be obtained from G_3. So, G_1 can be obtained from G_3 by a sequence of vertex deletions, edge deletions, and edge contractions. So, by Lemma 2.1, we have that $G_1 \preceq_m G_3$. $\square$

We remark that Lemma 2.2 can also be proven directly, using the definition of a minor.[14]

Lemma 2.3. *For all graphs G and H, if $H \preceq_t G$, then $H \preceq_m G$.*

Proof. Fix graphs G and H, and assume that $H \preceq_t G$. Then G contains a subgraph G' that is isomorphic to a subdivision of H, and clearly, H can be obtained from the subgraph G' by a sequence of edge contractions. Now Lemma 2.1 guarantees that $H \preceq_m G$. $\square$

Note that the converse of Lemma 2.3 is false, i.e. it is possible that $H \preceq_m G$, but $H \not\preceq_t G$. For example, the graph below contains $K_{1,4}$ as a minor (the branch sets are in dotted rectangles), but not as a topological minor (this is because $K_{1,4}$ contains a vertex of degree four, whereas the maximum degree in the graph below is three).



[13] Let us prove this fully formally. Suppose that G_2 is obtained from G_1 by first contracting an edge xy to a vertex v_{xy}, and then deleting a vertex z. If $z = v_{xy}$, then $G_2 = G_1 \setminus \{x, y\}$; otherwise, G_2 can be obtained from G_1 by first deleting z, and then contracting xy. Suppose now that G_2 is obtained from G_1 by first contracting an edge xy to a vertex v_{xy}, and then deleting an edge e. If v_{xy} is an endpoint of e, say $e = uv_{xy}$, then we can obtain G_2 from G_1 by first deleting all edges between u and $\{x, y\}$ (there is at least one and at most two such edges) and then contracting xy; otherwise, we can obtain G_2 from G_1 by first deleting e and then contracting xy.

[14] Proof?

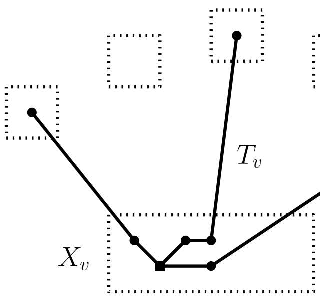
We do, however, have the following lemma.

Lemma 2.4. *Let G and H be graphs such that $H \preceq_m G$ and $\Delta(H) \leq 3$. Then $H \preceq_t G$.*

Proof. Let G' be a minimal subgraph of G such that $H \preceq_m G'$,[15] and let $\{X_v\}_{v \in V(H)}$ be the corresponding branch sets in $V(G')$. Our goal is to show that G' is itself a subdivision of H. By the minimality of G', we know that for all distinct $u, v \in V(H)$, we have that

- if $uv \in E(H)$, then there is exactly one edge between X_u and X_v in G',[16]
- if $uv \notin E(H)$, then there are no edges between X_u and X_v.[17]

By the minimality of G', $G'[X_v]$ is a tree.[18] Now, for each $v \in V(H)$, we let T_v be the graph obtained from $G'[X_v]$ by adding to it the edges between X_v and $V(G') \setminus X_v$ (and the endpoints of those edges); see the picture below.



Clearly, for each $v \in V(H)$, the graph T_v is a tree. Since $\Delta(H) \leq 3$, the minimality of G' guarantees that T_v has at most three leaves, and so $\Delta(T_v) \leq 3$. Moreover, T_v has at most one vertex of degree three, and if this vertex exists, then it belongs to X_v. Now, for all $v \in V(H)$, we let v' be the unique vertex of T_v of degree three if such a vertex exists, and otherwise, we let v' be any vertex in X_v. It is now clear that G' is a subdivision of H (vertices v' are the branch vertices), and so $H \preceq_t G$. $\square$

Lemma 2.5. *Let G be a graph. Then the following are equivalent:*

(1) G contains at least one $K_5, K_{3,3}$ as a topological minor;

(2) G contains at least one $K_5, K_{3,3}$ as a minor.

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[15] So, $H \preceq_m G'$, but for all proper subgraphs G'' of G', we have that $H \npreceq_m G''$.

[16] By the definition of a minor, there is at least one edge between X_u and X_v. If there is more than one such edge, then we can contradict the minimality of G' by deleting some edge between X_u and X_v.

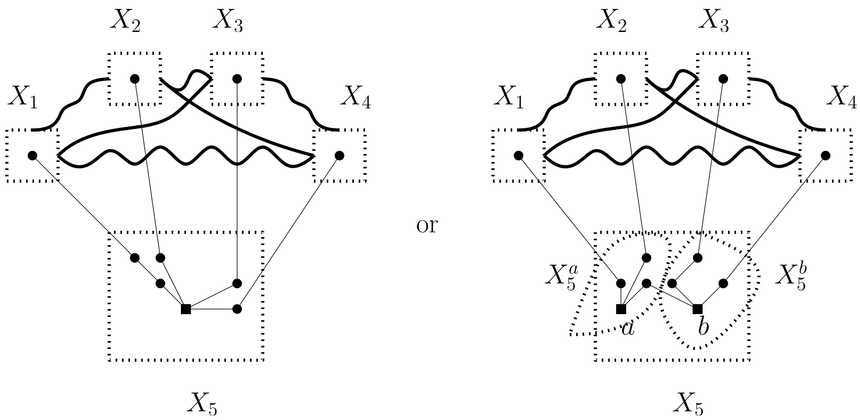
[17] Otherwise, we can contradict the minimality of G' by deleting an edge between X_u and X_v.

[18] Indeed, $G'[X_v]$ is connected, and therefore has a spanning tree, call it T. If $G'[X_v] \neq T$, then we can contradict the minimality of G' by deleting all edges in $E(G'[X_v]) \setminus E(T)$.

Proof. By Lemma 2.3, (1) implies (2). Suppose now that (2) holds. If $K_{3,3} \preceq_t G$, then Lemma 2.4 implies that $K_{3,3} \preceq_t K_{3,3}$, and we are done. Suppose now that $K_5 \preceq_m G$. Our goal is to show that either $K_5 \preceq_t G$ or $K_{3,3} \preceq_m G$.[19]

Let G' be a minimal subgraph of G such that $K_5 \preceq_m G'$. Let $X_1, \dots, X_5$ be the branch sets of the K_5 minor in G'.[20] By the minimality of G', we have that $G'[X_1], \dots, G'[X_5]$ are all trees, and for all distinct $i, j \in \{1, \dots, 5\}$, there is exactly one edge between X_i and X_j in G'. For each $i \in \{1, \dots, 5\}$, let T_i be the graph obtained from $G'[X_i]$ by adding the edges between X_i and $V(G') \setminus X_i$ (and the endpoints of those edges). Then for each $i \in \{1, \dots, 5\}$, T_i is a tree with exactly four leaves (each one of $X_1, \dots, X_5$, other than X_i, contains exactly one of those four leaves), and we deduce that T_i is a subdivision of one of the following two trees.

If $T_1, \dots, T_5$ are all subdivisions of $K_{1,4}$ (see the picture below, on the left), then it is clear that G' is a subdivision of K_5, and it follows that $K_5 \preceq_t G$. Suppose now that at least one of $T_1, \dots, T_5$ is a subdivision of T (see the picture below, on the right); by symmetry, we may assume that T_5 is a subdivision of T, and we let a, b be the two vertices of T_5 of degree three (note that $a, b \in X_5$). Now let X_5^a be the set of all vertices $v \in X_5$ such that the (unique) path between v and a in the tree T_5 does not contain the vertex b, and let $X_5^b := X_5 \setminus X_5^a$. Then $a \in X_5^a$ and $b \in X_5^b$, and it is easy to see that G contains a $K_{3,3}$ minor with branch sets $X_1, \dots, X_4, X_5^a, X_5^b$. But now Lemma 2.4 implies that $K_{3,3} \preceq G$, and we are done.



[19]Note that this is enough. Indeed, if $K_5 \preceq_t G$, then we are done. And if $K_{3,3} \preceq_m G$, then Lemma 2.4 guarantees that $K_{3,3} \preceq_t G$, and again we are done.

[20]So, $G'[X_1], \dots, G'[X_5]$ are connected, and for all distinct $i, j \in \{1, \dots, 5\}$, there is an edge between X_i and X_j in G'.

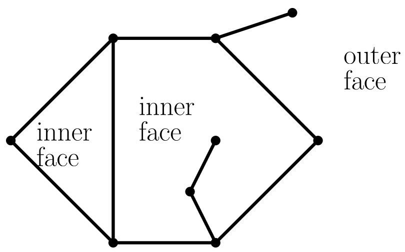
□

3 Planar graphs

A graph is *planar* if it can be drawn in the plane without any edge crossings.

Obviously, a graph can be drawn in the plane without any edge crossings if and only if it can be drawn on a sphere without any edge crossings. So, planar graphs are precisely those that can be drawn on a sphere without any edge crossings.

When we draw a graph on a plane without edge crossings, we divide the plane into regions, called *faces*; one of the faces, called the *outer face* is unbounded, and the remaining faces (called *inner faces*) are bounded.



We can define faces on a sphere analogously, but in this case, all faces are bounded, and we get no asymmetry between the inner faces and the outer face. For this reason, for the purposes of proving theorems, it is often more practical to draw on a sphere than on a plane.

Lemma 3.1. *If a graph is planar, then so are all its minors.*

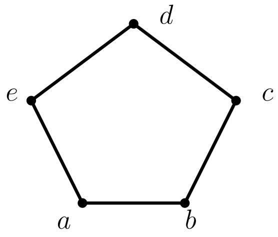
Proof. Clearly, any graph obtained from a planar graph by deleting one vertex, deleting one edge, or contracting one edge is planar. So, by Lemma 2.1, all minors of a planar graph are planar. □

A *homeomorphism* of the sphere is a bijection f from the sphere to itself such that both f and f^{-1} are continuous. Informally, a homeomorphism of the sphere is the result of "stretching" the sphere (and possibly also rotating and taking mirror images).

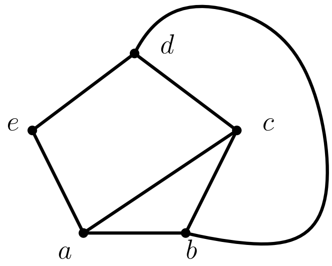
Two graph drawings on the sphere are *equivalent* if some sphere homeomorphism transforms one drawing into the other.

Lemma 3.2. *Graphs K_5 and $K_{3,3}$ are not planar. Consequently, no planar graph contains K_5 or $K_{3,3}$ as a minor.*

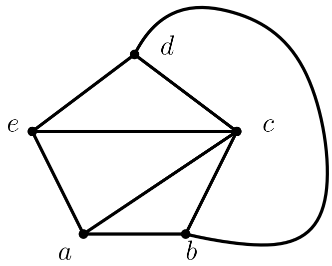
Proof. Suppose that K_5 is planar, so that we can draw it on a sphere without any edge crossings. Let $\{a, b, c, d, e\}$ be the vertex set of the K_5. We first draw the 5-cycle a, b, c, d, e, a on the sphere.



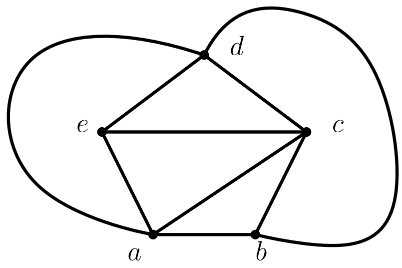
Since edges ac and bd do not cross, we must draw them through distinct faces created by our 5-cycle a, b, c, d, e, a, and we obtain the following.[21]



There is now only one way to add the edge ce to our drawing without creating edge crossings, as shown below.



Further, there is only one way to add the edge ad to our drawing without creating edge crossings, as shown below.



But now it is not possible to add the edge be to our drawing without creating edge crossings. So, K_5 is not planar.

A similar argument shows that $K_{3,3}$ is not planar.[22]

[21] Remember, we are on a sphere! So, we have full symmetry between the two faces produced by the 5-cycle a, b, c, d, e, a.

[22] Check this!

Since K_5 and $K_{3,3}$ are not planar, Lemma 3.1 guarantees that no planar graph contains K_5 or $K_{3,3}$ as a minor. $\square$

The following theorem is usually referred to as "Kuratowski's theorem," or sometimes as the "Kuratowski-Wagner theorem."

Theorem 3.3 (Kuratowski, 1930; Wagner, 1937)**.** *Let G be a graph. Then the following are equivalent:*

(a) G is planar;

(b) G contains neither K_5 nor $K_{3,3}$ as a minor;

(c) G contains neither K_5 nor $K_{3,3}$ as a topological minor.

We have already proven the "easy" part of Kuratowski's theorem: (a) implies (b) by Lemma 3.2, and (b) is equivalent to (c) by Lemma 2.5. It remains to prove the "hard" part: (b) implies (a). We will do this in the next lecture.