NDMI012: Combinatorics and Graph Theory 2

Lecture #2

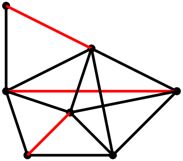
Edmonds' Blossom algorithm

Irena Penev

February 22, 2022

Definition

A *matching* in a graph G is a collection of edges of G, no two of which share an endpoint.



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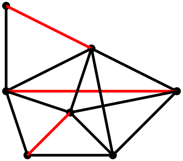

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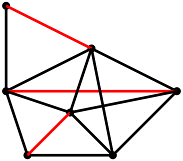

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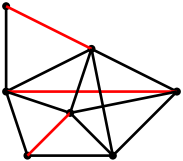
- Our goal is to describe a polynomial-time algorithm that finds a maximum matching in a graph.



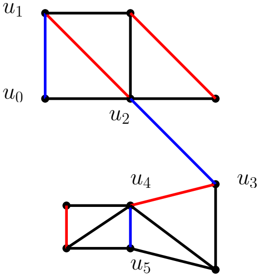
Definition

Let M be a matching and v a vertex of G. If v is incident with some edge of M, then v is *saturated* by M. Otherwise, v is *unsaturated* by M.

Definition

Let M be a matching in a graph G. An *M-alternating path* is a path $u_0, u_1, \ldots, u_t$ in G s.t. every other edge of the path belongs to M (and the remaining edges do not). An *M-augmenting path* is an M-alternating path $u_0, u_1, \ldots, u_t$ ($t \neq 0$) s.t. u_0, u_t are both unsaturated by M.

- For instance, in the picture below, $u_0, u_1, u_2, u_3, u_4, u_5$ is an M-augmenting path (edges of the matching M are in red).



Lemma 1.1

Let M be a matching in a graph G, and let $u_0, u_1, \ldots, u_t$ be an M-augmenting path. Then t is odd and

$$\begin{aligned} M' \quad := \quad & \Big(M \setminus \{u_1u_2, u_3u_4, \ldots, u_{t-2}u_{t-1}\}\Big) \\ & \cup \{u_0u_1, u_2u_3, \ldots, u_{t-1}u_t\} \end{aligned}$$

is a matching of G satisfying $|M'| = |M| + 1$.

Proof. This follows from the relevant definitions.

Theorem 1.2 [Berge, 1957]

Let M be a matching in a graph G. Then M is a maximum matching of G iff G has no M-augmenting path.

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Clearly, $\Delta(H) \leq 2$.

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Clearly, $\Delta(H) \leq 2$. So, H is the disjoint union of paths and cycles.

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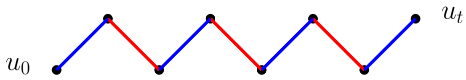
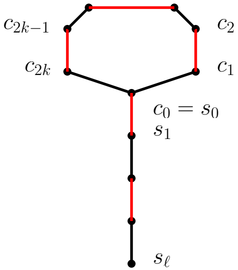
Proof (continued). Now, since $|M'| > |M|$, some component P of H has more edges of M' than of M. If P is a cycle, then we see that some vertex of P is incident two edges of M', contrary to the fact that M' is a matching. So, P is a path, and it is easy to see that it is in fact an M-augmenting path in G.



Definition

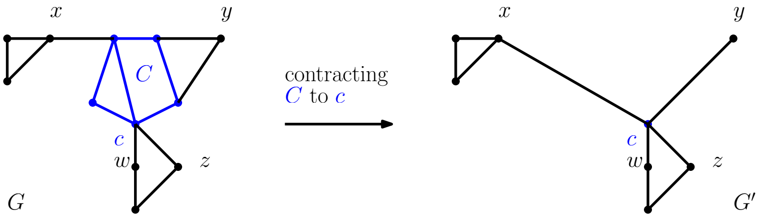
Suppose that M is a matching in a graph G. A *blossom* is a cycle $c_0, c_1, \ldots, c_{2k}, c_0$ of length $2k+1$ (with $k \geq 1$) in G in which edges $c_1c_2, c_3c_4, \ldots, c_{2k-1}c_{2k}$ belong to M, and the remaining $k+1$ edges do not belong to M. A *stem* for this blossom is an M-alternating path $s_0, \ldots, s_\ell$ of even length s.t. $s_0 = c_0$ is the unique common vertex of the cycle $c_0, c_1, \ldots, c_{2k}, c_0$ and the path $s_0, \ldots, s_\ell$, and s_ℓ is unsaturated by M. A *flower* is the union of a blossom and a corresponding stem.



Definition

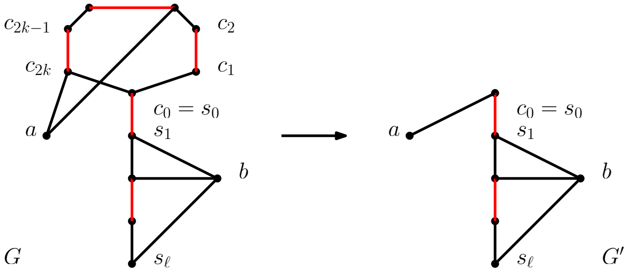
Let G be a graph, and let $C \subseteq V(G)$ and $c \in C$. We say that G' is the graph obtained form G by *contracting* C to c if

- $V(G') = V(G) \setminus (C \setminus \{c\}) = (V(G) \setminus C) \cup \{c\}$, and
- $E(G') = \left(\binom{V(G)\setminus C}{2} \cap E(G)\right) \cup \left\{xc \mid x \in V(G) \setminus C,\ \exists c' \in C \text{ s.t. } xc' \in E(G)\right\}$.



Lemma 2.1

Let M be a matching in a graph G, and let $C = c_0, \ldots, c_{2k}, c_0$ be a blossom and $S = s_0, \ldots, s_\ell$ a corresponding stem (in particular, $c_0 = s_0$). Let G' be the graph obtained from G by contracting C to c_0, and let $M' = M \setminus E(C)$. Then M' is a matching of G'. Furthermore, M is a maximum matching of G if and only if M' is a maximum matching of G'.



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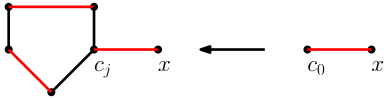
Suppose now that c_0 is saturated by M''. Then there exists some vertex $x \in V(G) \setminus V(C)$ and an index $j \in \{0, \dots, 2k\}$ s.t. $xc_j \in E(G)$. But now the matching $(M'' \setminus \{xc_0\}) \cup \{xc_j\} \cup \{c_{j+1}c_{j+2}, c_{j+3}c_{j+4}, \dots, c_{j+2k-1}c_{j+2k}\}$ is a matching of G of size greater than $|M|$.



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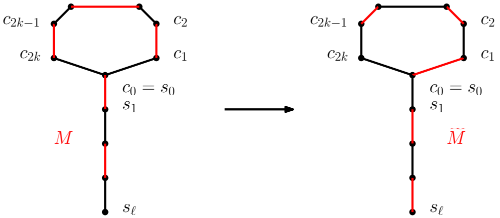

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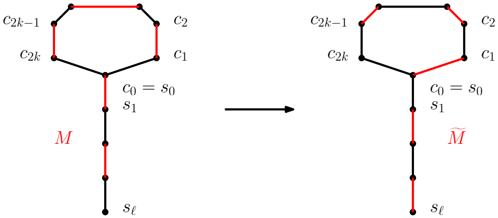

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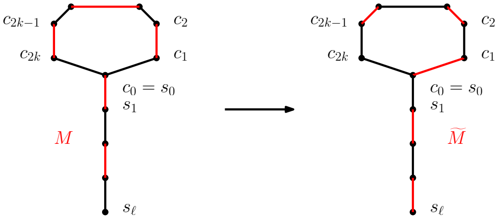

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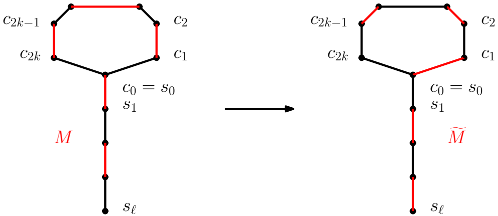

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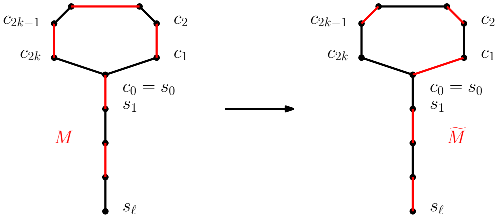

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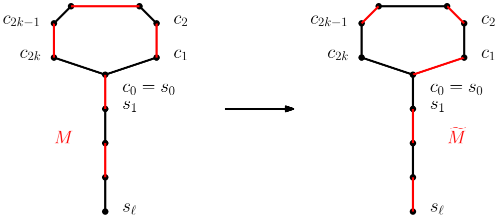

If $V(P) \cap V(C) = \emptyset$, then P is an $\widetilde{M}'$-augmenting path in G', and we are done.

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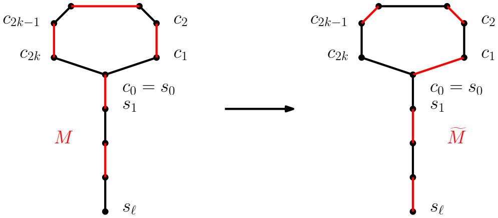

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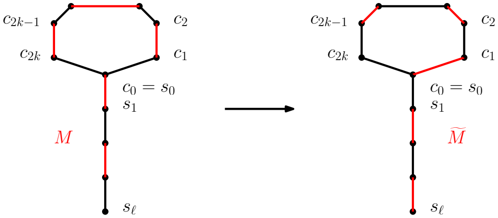

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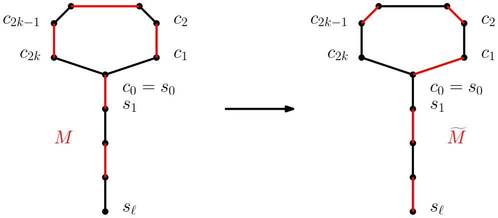

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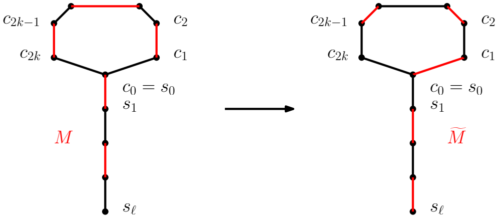

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Edmonds' Blossom algorithm

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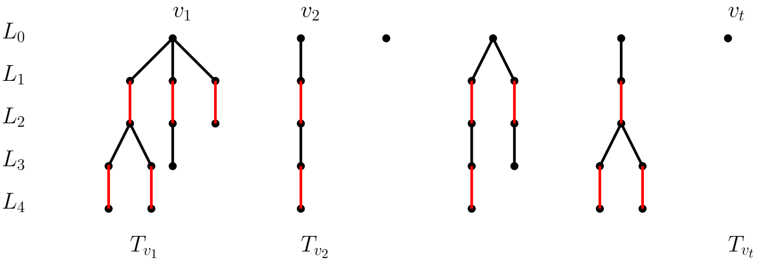
  - We proceed as follows.

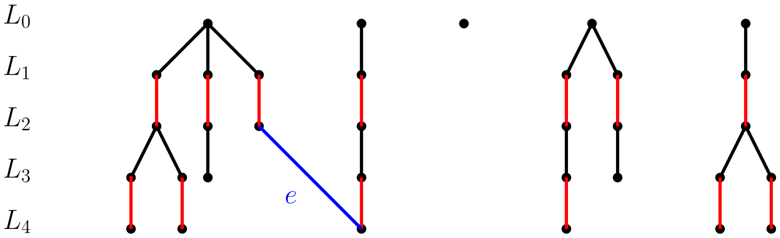
Edmonds' Blossom algorithm

- **Step 1.** First, using breadth-first search, we form an auxiliary forest F (which is a subgraph of G) as follows. $V(F)$ is partitioned into levels, $L_0, L_1, L_2, \dots,$ where:
  - level L_0 consists of all vertices of G that are unsaturated by M;
  - for all integers $k \geq 0$, L_k is the set of vertices at distance k (in F) from L_0;
  - for an even integers $k \geq 0$, edges between L_k and L_{k+1} in F do not belong to M, and edges between L_{k+1} and L_{k+2} in F do belong to M.



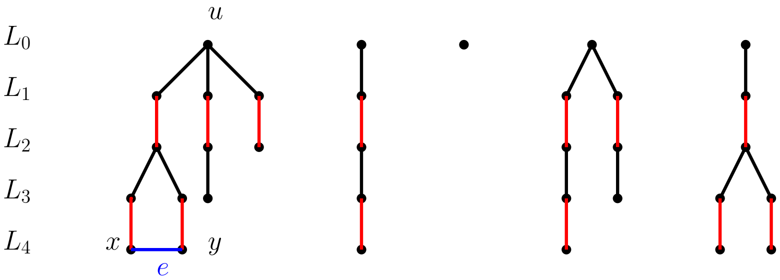
Edmonds' Blossom algorithm

- **Step 2.** If there exists an edge $e \in E(G)$ between even levels of two distinct trees, we obtain an M-augmenting path, and then we obtain a matching of size $|M| + 1$, as in Lemma 1.1.



Edmonds' Blossom algorithm

- **Step 2 (continued).** If there exists an edge $e \in E(G)$ between two vertices, say x and y, belonging to even levels of the same tree T_u, then we can find a flower (i.e. a blossom with a corresponding stem).



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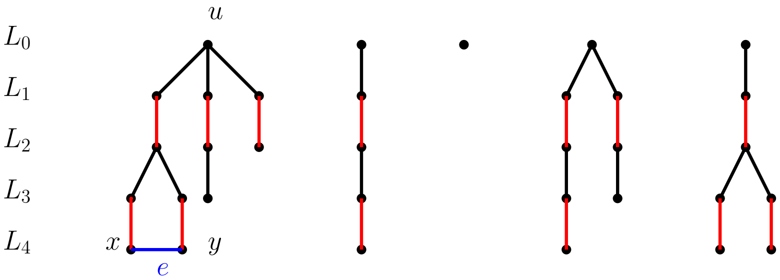

  - Let G' be the graph obtained from G by contracting C to a vertex c_0, and let $M' = M \setminus E(C)$ (as in Lemma 2.1).

Edmonds' Blossom algorithm

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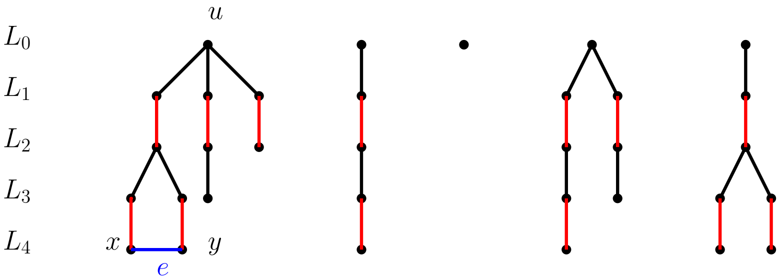

  - Let G' be the graph obtained from G by contracting C to a vertex c_0, and let $M' = M \setminus E(C)$ (as in Lemma 2.1).
  - We now call the algorithm with input G' and M'.

Edmonds' Blossom algorithm

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Edmonds' Blossom algorithm

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  - If we obtain the answer that M' is a maximum matching in G', then (by Lemma 2.1) M is a maximum matching in G, and we are done.

Edmonds' Blossom algorithm

- **Step 2 (continued).** Then there are two cases.
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  - Suppose we obtained a matching M'' in G' that is size greater than $|M'|$.

Edmonds' Blossom algorithm

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Edmonds' Blossom algorithm

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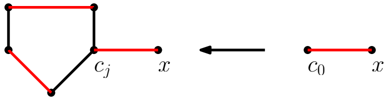
    - If c_0 is saturated by M'', then we can obtain a matching of G of size greater than $|M|$ as in the proof of Lemma 2.1.

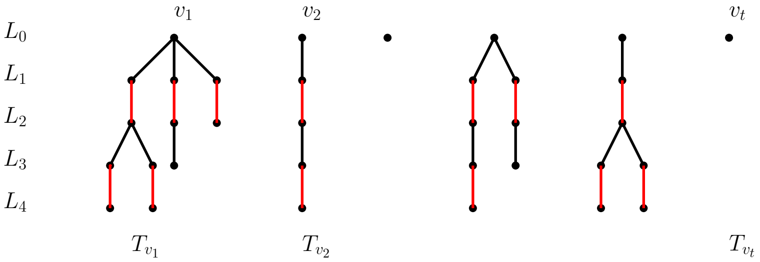


Edmonds' Blossom algorithm

- **Step 2 (continued).**
  - If there is an edge $e \in M \setminus E(F)$ has at least one endpoint in $V(F)$, then we get either a flower or an M-augmenting path (this is similar to the above; details: Lecture Notes).

Edmonds' Blossom algorithm

- **Step 2 (continued).**
  - If there is an edge $e \in M \setminus E(F)$ has at least one endpoint in $V(F)$, then we get either a flower or an M-augmenting path (this is similar to the above; details: Lecture Notes).
  - Suppose now that there are no edges (of G) between vertices in even levels, and moreover, that every edge of M that has an endpoint in $V(F)$ is in fact an edge of F. In this case, G contains no M-augmenting path (details: Lecture Notes) and so by Theorem 1.2, M is a maximum matching of G.



- **Remark:** The running time of Edmonds' Blossom algorithm is $O(n^4)$, if the algorithm is implemented in the obvious way.