NDMI012: Combinatorics and Graph Theory 2

Lecture #1

Matchings in general graphs

Irena Penev

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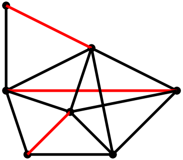
Definition

A *matching* in a graph G is a collection of edges of G, no two of which share an endpoint.



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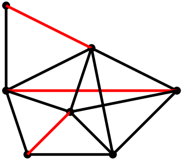

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A *maximum matching* of G is a matching M of G s.t. for all matchings M' of G, we have that $|M'| \leq |M|$. The *matching number* of G, denoted by $\nu(G)$, is the size of a maximum matching (i.e. the number of edges in a maximum matching).

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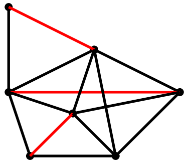

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- Remark: $\nu(G) \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor$.

Definition

If M is a matching and v is a vertex of a graph G, then we say that v is *saturated* by M if v is incident with some edge of M. A set $X \subseteq V(G)$ is *saturated* by M if every vertex in X is saturated by M.

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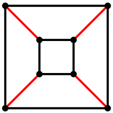

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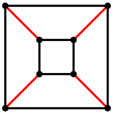

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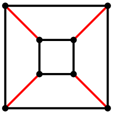
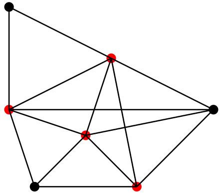


- A graph G has a perfect matching iff $\nu(G) = \frac{|V(G)|}{2}$.
- In particular, every graph that has a perfect matching, has an even number of vertices.

- In Combinatorics & Graphs 1, we proved a couple of theorems about matchings in bipartite graphs:
  - the Kőnig-Egerváry theorem;
  - Hall's theorem.
- Here, we state both these theorems without proof.
- We will use Hall's theorem later in the lecture.

Definition

A *vertex cover* of a graph G is any set C of vertices of G such that every edge of G has at least one endpoint in C.



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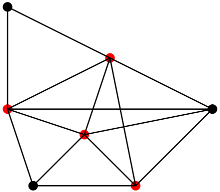

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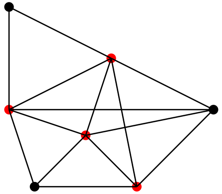

- For any graph G, any vertex cover of G is of size $\geq \nu(G)$.
  - Indeed, if C is a vertex cover of G, and M is a matching of G, then C contains at least one endpoint of each edge of M; since no two edges of M share an endpoint, it follows that $|C| \geq |M|$.

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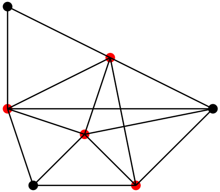


The Kőnig-Egerváry theorem

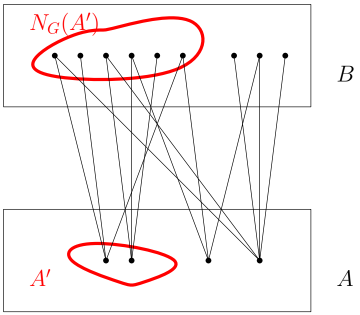
The maximum size of a matching in a bipartite graph is equal to the minimum size of a vertex cover in that graph.

Hall's theorem

Let G be a bipartite graph with bipartition (A, B). Then the following are equivalent:

(a) all sets $A' \subseteq A$ satisfy $|A'| \leq |N_G(A')|$;

(b) G has an A-saturating matching.



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The Tutte-Berge formula

Every graph G satisfies
$\nu(G) = \frac{1}{2} \min_{U \subseteq V(G)} \Big(|V(G)| + |U| - \mathrm{odd}(G \setminus U) \Big).$

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- Then, we give an application of Tutte's theorem (called Petersen's theorem).

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- Then, we give an application of Tutte's theorem (called Petersen's theorem).
- Finally, we prove the Tutte-Berge formula.

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Proof. Fix a graph G. Clearly, the following are equivalent:

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By the Tutte-Berge formula, (b) is equivalent to

(c) $\nu(G) \geq \frac{|V(G)|}{2}$.

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But clearly, (c) holds iff G has a perfect matching. So, (a) holds iff G has a perfect matching, which is what we needed to show.

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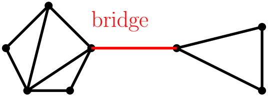
A *bridge* in a graph G is an edge $e \in E(G)$ s.t. $G - e$ has more components than G. A graph is *bridgeless* if it has no bridge.



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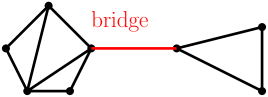

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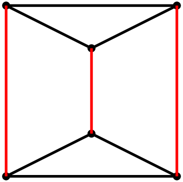

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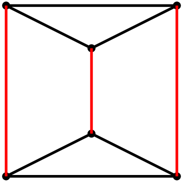

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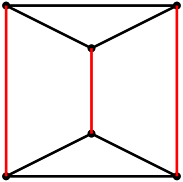

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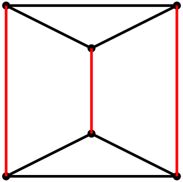

Proof. Fix a cubic, bridgeless graph G. We will apply Tutte's theorem. Fix $S \subseteq V(G)$; we must show that $|S| \geq \text{odd}(G \setminus S)$.

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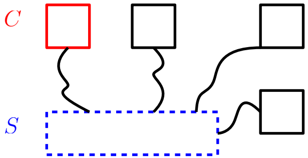

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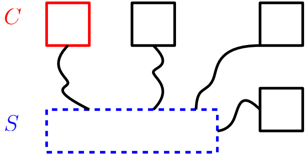

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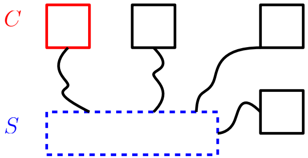

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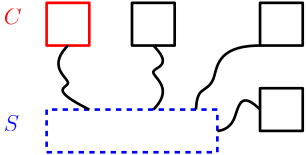

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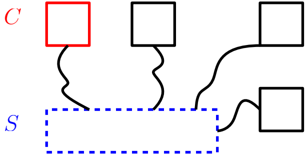

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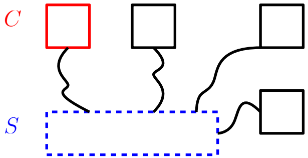

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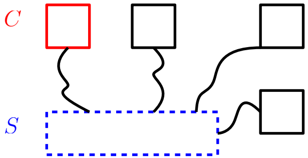

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A graph G has a perfect matching iff every set $S \subseteq V(G)$ satisfies $|S| \geq \text{odd}(G \setminus S)$.

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Every cubic, bridgeless graph has a perfect matching.

Proof (continued). Reminder: G is cubic and bridgeless, and $S \subseteq V(G)$. WTS $|S| \geq \text{odd}(G \setminus S)$.

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Set $t := \text{odd}(G \setminus S)$. By the Claim, the number of edges between S and $V(G) \setminus S$ is at least $3t$. On the other hand, since G is cubic, the total number of edges incident with at least one vertex of S as at most $3|S|$. Thus, $3t \leq 3|S|$, i.e. $|S| \geq t = \text{odd}(G \setminus S)$.

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Proof (continued). Reminder: G is cubic and bridgeless, and $S \subseteq V(G)$. WTS $|S| \geq \text{odd}(G \setminus S)$.

> ***Claim.*** *For all odd components C of $G \setminus S$, there are at least three edges between S and $V(C)$ in G.*

Set $t := \text{odd}(G \setminus S)$. By the Claim, the number of edges between S and $V(G) \setminus S$ is at least $3t$. On the other hand, since G is cubic, the total number of edges incident with at least one vertex of S as at most $3|S|$. Thus, $3t \leq 3|S|$, i.e. $|S| \geq t = \text{odd}(G \setminus S)$. So, by Tutte's theorem, G has a perfect matching.

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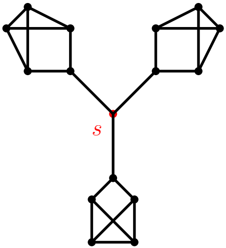

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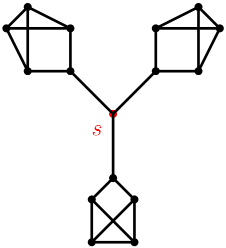

- The graph above (call it G) is cubic, but not bridgeless. For $S := \{s\}$, we have $\text{odd}(G \setminus S) = 3$, and so $|S| < \text{odd}(G \setminus S)$.
- Thus, by Tutte's theorem, G does not have a perfect matching.

- It remains to prove the Tutte-Berge formula!

The Tutte-Berge formula

Every graph G satisfies
$\nu(G) = \frac{1}{2} \min_{U \subseteq V(G)} \Big(|V(G)| + |U| - \mathrm{odd}(G \setminus U) \Big)$.

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Let G be a graph. Then for all $S \subseteq V(G)$, we have that
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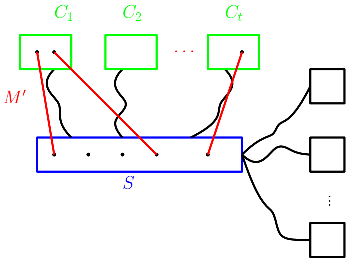
Proof. Fix $S \subseteq V(G)$, set $t := \text{odd}(G \setminus S)$, and let $C_1, \dots, C_t$ be the odd components of $G \setminus S$. Fix any matching M in G. Let M' be the set of all edges of M that have one endpoint in S and the other one in $V(C_1) \cup \dots V(C_t)$; obviously, $|M'| \leq |S|$.



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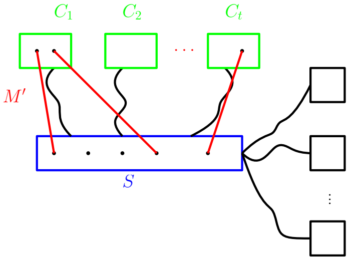

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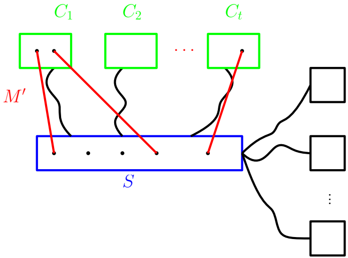

Since the components $C_1, \ldots, C_t$ are all odd, at least $t - |M'| \geq t - |S|$ of them have a vertex that is unsaturated by M.

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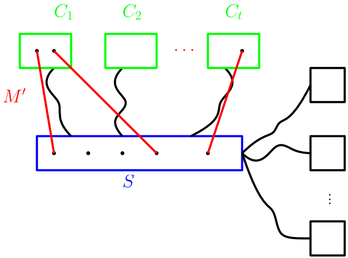

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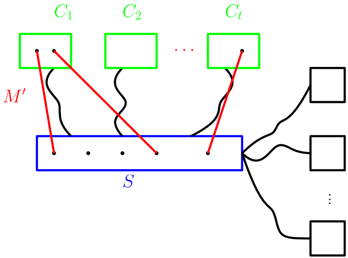
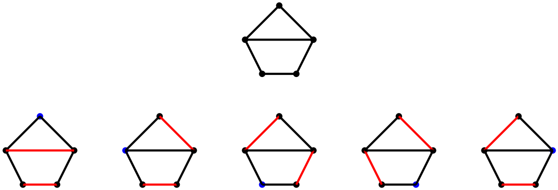


Since the components $C_1, \ldots, C_t$ are all odd, at least $t - |M'| \geq t - |S|$ of them have a vertex that is unsaturated by M. So, the total number of vertices of G that are saturated by M is at most $|V(G)| + |S| - t$, and so $|M| \leq \frac{|V(G)|+|S|-t}{2}$. Thus, $\nu(G) \leq \frac{|V(G)|+|S|-t}{2} = \frac{|V(G)|+|S|-\text{odd}(G\backslash S)}{2}$.

Definition

A graph G is *hypomatchable* if it does not have a perfect matching, but for all $v \in V(G)$, the graph $G \setminus v$ does have a perfect matching.



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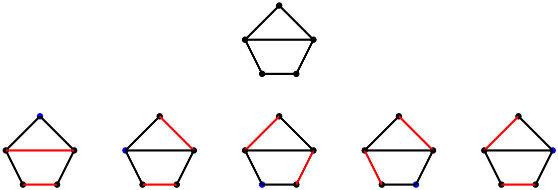

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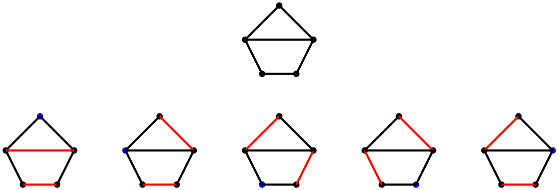
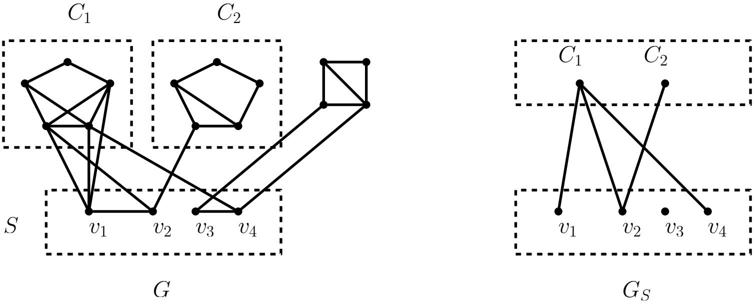


- A *hypomatchable component* of a graph G is a component of G that is a hypomatchable graph.
- Obviously, every hypomatchable component of G is odd.

- For a graph G and a set $S \subseteq V(G)$, let us denote by G_S the bipartite graph whose one side of the bipartition is S, and whose other side of the bipartition is the collection of all odd components of $G \setminus S$, and in which a vertex $v \in S$ and an odd component C of $G \setminus S$ are adjacent iff v has a neighbor in $V(C)$ in G.



Definition

A *Gallai-Edmonds set* in a graph G is a set $S \subseteq V(G)$ s.t.

- every component of $G \setminus S$ is hypomatchable;
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If S is a Gallai-Edmonds set of a graph G, then $\nu(G) = \frac{|V(G)|+|S|-\text{odd}(G\setminus S)}{2}$.

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- But first, let us show that they (together) imply the Tutte-Berge formula.

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Every graph G satisfies
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To simplify notation, set $n := |V(G)|$, $s := |S|$, and $t := \text{odd}(G \setminus S)$. We must show that $\nu(G) \geq \frac{n+s-t}{2}$.

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Let $C_1, \ldots, C_t$ be the odd components of $G \setminus S$ (since all components of $G \setminus S$ are hypomatchable and therefore odd, we see that $C_1, \ldots, C_t$ are in fact all the components of $G \setminus S$), and set $S = \{v_1, \ldots, v_s\}$.

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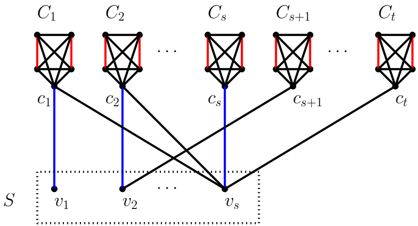

Since S is a Gallai-Edmonds set, G_S has an S-saturating matching, call it M_S. By symmetry, WMA $M_S = \{v_1 C_1, \ldots, v_s C_s\}$. For each $i \in \{1, \ldots, s\}$, choose a vertex $c_i \in V(C_i)$ s.t. $v_i c_i \in E(G)$. For all $i \in \{s+1, \ldots, t\}$, choose any vertex $c_i \in C_i$.

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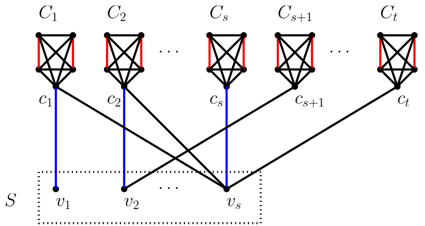
Proof (continued).



Further, since S is a Gallai-Edmonds set, for all $i \in \{1, \ldots, t\}$, C_i is hypomatchable, and in particular, $C_i \setminus c_i$ has a perfect matching, call it M_i. Now, set $M := \{v_1c_1, \ldots, v_sc_s\} \cup M_1 \cup \cdots \cup M_t$.

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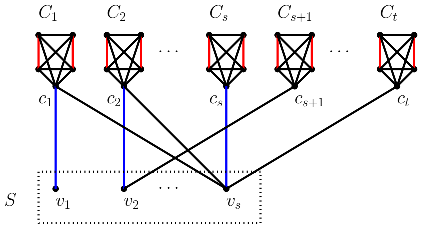

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Claim 1. *All components of $G \setminus S$ are odd.*

Proof of Claim 1 (outline).

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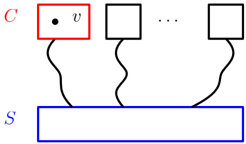

Fix $v \in V(C)$, and set $S' := S \cup \{v\}$. Then S' contradicts the choice of S (details: Lecture Notes). This proves Claim 1.

Lemma 3.3

Every graph has a Gallai-Edmonds set.

Proof. Reminder: $\text{odd}(G \setminus S) - |S|$ is as large as possible, and subject to that, $|S|$ is as large as possible.

Claim 2. *All components of $G \setminus S$ are hypomatchable.*

Proof of Claim 2 (outline).

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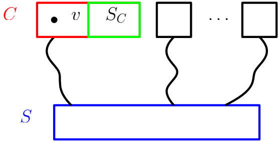
Claim 2. *All components of $G \setminus S$ are hypomatchable.*

Proof of Claim 2 (outline). Suppose otherwise, and fix a component C of $G \setminus S$ and a vertex $v \in V(C)$ s.t. $C \setminus v$ does not have a perfect matching. By Claim 1, $C \setminus v$ has an even number of vertices; since $C \setminus v$ does not have a perfect matching, it follows that $\nu(C \setminus v) \leq \frac{|V(C) \setminus \{v\}|}{2} - 1 = \frac{|V(C)|-3}{2}$. By the induction hypothesis, $C \setminus v$ has a Gallai-Edmonds set, call it S_C.



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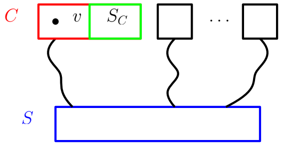
Proof of Claim 2 (outline, continued).



$$\begin{aligned}
\frac{|V(C)|-3}{2} &\geq \nu(C \setminus v) \\
&= \frac{|V(C\setminus v)|+|S_C|-\text{odd}\Big((C\setminus v)\setminus S_C\Big)}{2} \qquad \text{by Lemma 3.2} \\
&= \frac{|V(C)|-1+|S_C|-\text{odd}\Big((C\setminus v)\setminus S_C\Big)}{2},
\end{aligned}$$

and so $\text{odd}\Big((C \setminus v) \setminus S_C\Big) \geq |S_C| + 2$.

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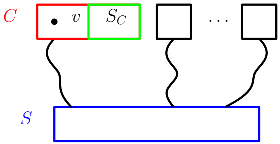

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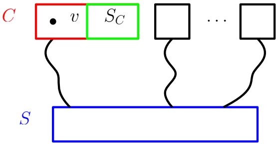


Now, set $S' := S \cup \{v\} \cup S_C$. Then

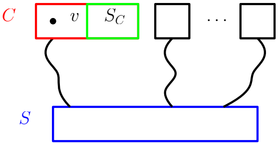
$$\begin{aligned}
\mathrm{odd}(G \setminus S') &= \mathrm{odd}(G \setminus S) - 1 + \mathrm{odd}\Big((C \setminus v) \setminus S_C\Big) \\
&\geq \mathrm{odd}(G \setminus S) - 1 + (|S_C| + 2) \\
&= \mathrm{odd}(G \setminus S) + |S_C| + 1 \\
&= \mathrm{odd}(G \setminus S) + (|S'| - |S|),
\end{aligned}$$

and so $\mathrm{odd}(G \setminus S') - |S'| \geq \mathrm{odd}(G \setminus S) - |S|$.

Proof of Lemma 3.3. Reminder: $\text{odd}(G \setminus S) - |S|$ is as large as possible, and subject to that, $|S|$ is as large as possible.

Claim 2. *All components of $G \setminus S$ are hypomatchable.*

Proof of Claim 2 (outline, continued). Reminder: $\text{odd}(G \setminus S') - |S'| \geq \text{odd}(G \setminus S) - |S|$.



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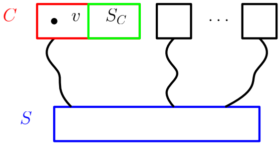

Since we also have that $|S'| > |S|$, this contradicts the choice of S. This proves Claim 2.

Lemma 3.3

Every graph has a Gallai-Edmonds set.

Proof of Lemma 3.3. Reminder: $\text{odd}(G \setminus S) - |S|$ is as large as possible, and subject to that, $|S|$ is as large as possible.

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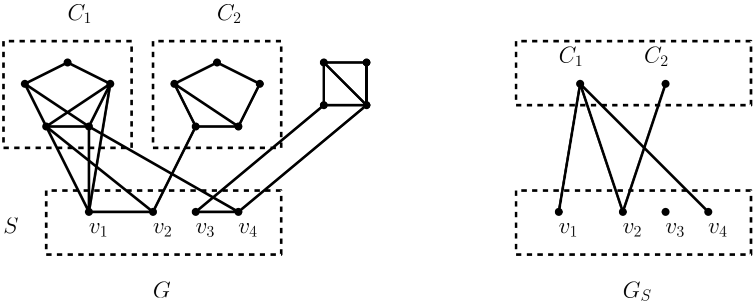
Claim 3. *G_S has an S-saturating matching.*

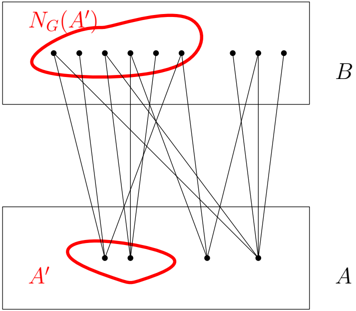


Hall's theorem

Let G be a bipartite graph with bipartition (A, B). Then the following are equivalent:

(a) all sets $A' \subseteq A$ satisfy $|A'| \leq |N_G(A')|$;

(b) G has an A-saturating matching.



Proof of Lemma 3.3. Reminder: $\text{odd}(G \setminus S) - |S|$ is as large as possible, and subject to that, $|S|$ is as large as possible.

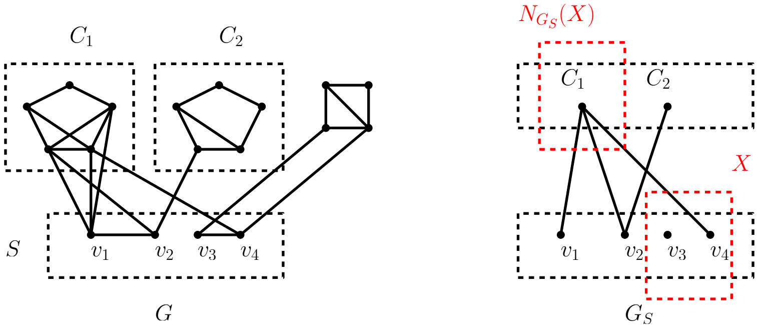
Claim 3. *G_S has an S-saturating matching.*

Proof of Claim 3.

Proof of Lemma 3.3. Reminder: $\mathrm{odd}(G \setminus S) - |S|$ is as large as possible, and subject to that, $|S|$ is as large as possible.

Claim 3. *G_S has an S-saturating matching.*

Proof of Claim 3. Suppose otherwise. Then by Hall's theorem, there exists a set $X \subseteq S$ s.t. $|N_{G_S}(X)| < |X|$.



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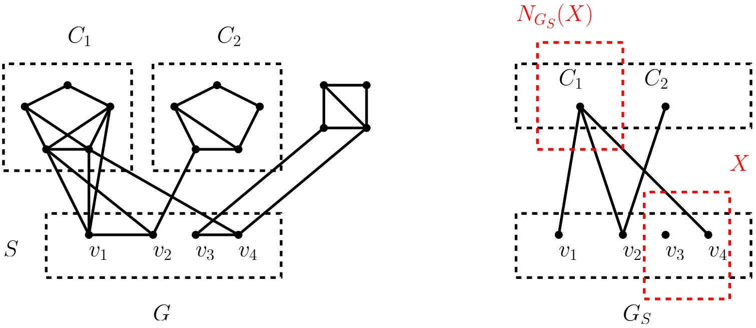
Claim 3. *G_S has an S-saturating matching.*

Proof of Claim 3. Suppose otherwise. Then by Hall's theorem, there exists a set $X \subseteq S$ s.t. $|N_{G_S}(X)| < |X|$. Set $S' := S \setminus X$.



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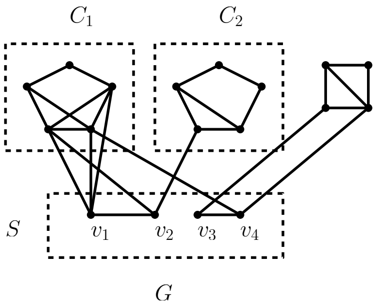

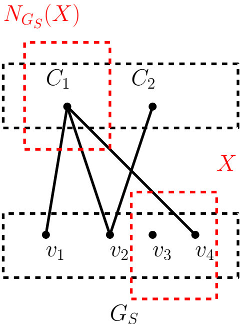

Then all odd components of $G \setminus S$ other than the ones in $N_{G_S}(X)$ are still odd components of $G \setminus S'$, and we compute:

Proof of Lemma 3.3. Reminder: $\text{odd}(G \setminus S) - |S|$ is as large as possible, and subject to that, $|S|$ is as large as possible.

Claim 3. *G_S has an S-saturating matching.*

Proof of Claim 3. Reminder: $|N_{G_S}(X)| < |X|$, $S' := S \setminus X$.

$$\begin{aligned}
\text{odd}(G \setminus S') &\geq \text{odd}(G \setminus S) - |N_{G_S}(X)| \\
&> \text{odd}(G \setminus S) - |X| \\
&= \text{odd}(G \setminus S) - (|S| - |S'|) \\
&= \text{odd}(G \setminus S) - |S| + |S'|,
\end{aligned}$$

Proof of Lemma 3.3. Reminder: $\mathrm{odd}(G \setminus S) - |S|$ is as large as possible, and subject to that, $|S|$ is as large as possible.

Claim 3. *G_S has an S-saturating matching.*

Proof of Claim 3. Reminder: $|N_{G_S}(X)| < |X|$, $S' := S \setminus X$.

$$\begin{aligned} \mathrm{odd}(G \setminus S') &\geq \mathrm{odd}(G \setminus S) - |N_{G_S}(X)| \\ &> \mathrm{odd}(G \setminus S) - |X| \\ &= \mathrm{odd}(G \setminus S) - (|S| - |S'|) \\ &= \mathrm{odd}(G \setminus S) - |S| + |S'|, \end{aligned}$$

and it follows that

$$\mathrm{odd}(G \setminus S') - |S'| > \mathrm{odd}(G \setminus S) - |S|,$$

contrary to the choice of S. This proves Claim 3.

Lemma 3.3

Every graph has a Gallai-Edmonds set.

Proof (continued). Reminder: $\mathrm{odd}(G \setminus S) - |S|$ is as large as possible, and subject to that, $|S|$ is as large as possible.

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Claim 2. *All components of $G \setminus S$ are hypomatchable.*

Claim 3. *G_S has an S-saturating matching.*

By Claims 2 and 3, we have that S is a Gallai-Edmonds set of G.

- We have proven the following three theorems.

The Tutte-Berge formula

Every graph G satisfies
$\nu(G) = \frac{1}{2} \min_{U \subseteq V(G)} \Big(|V(G)| + |U| - \mathrm{odd}(G \setminus U) \Big)$.

Tutte's theorem

A graph G has a perfect matching iff every set $S \subseteq V(G)$ satisfies $|S| \geq \mathrm{odd}(G \setminus S)$.

Petersen's theorem

Every cubic, bridgeless graph has a perfect matching.

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- A maximum matching can be found in polynomial time (Edmonds, 1961).
    - Details: Next time!