NDMI012: Combinatorics and Graph Theory 2

Lecture #1
Matchings in general graphs

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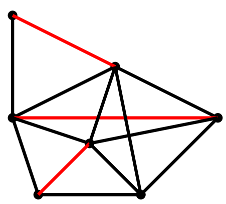
1 Basic notions

In this lecture, all graphs are finite and simple (i.e. have no loops and no parallel edges). For convenience, we will allow our graphs to possibly be null (i.e. have no vertices and no edges).

A *matching* in a graph G is a collection of edges of G, no two of which share an endpoint. An example of a matching is shown below (the edges of the matching are in red).



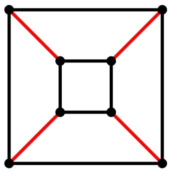
A *maximum matching* of G is a matching M of G such that for all matchings M' of G, we have that $|M'| \leq |M|$. The *matching number* of G, denoted by $\nu(G)$, is the size of a maximum matching (i.e. the number of edges in a maximum matching).[1] Trivially, $\nu(G) \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor$.

If M is a matching and v is a vertex of a graph G, then we say that v is *saturated* by M (or that M *saturates* v) provided that v is incident with some edge of M. If M does not saturate v, then v is *unsaturated* by M. A set $X \subseteq V(G)$ is *saturated* by M if every vertex in X is saturated by M.

A matching M of a graph graph G is *perfect* if all vertices of G are saturated by M. Obviously, a graph G has a perfect matching if and only if $\nu(G) = \frac{|V(G)|}{2}$. In particular, every graph that has a perfect matching, has

[1] So, $\nu(G) = \max\{|M| \mid M \text{ is a matching of } G\}$.

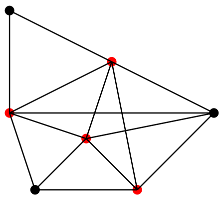
an even number of vertices.[2] An example of a perfect matching is shown below (the edges of the matching are in red).



2 Matchings in bipartite graphs

In Combinatorics & Graphs 1, we proved a couple of theorems about matchings in bipartite graphs: the Kőnig-Egerváry theorem and Hall's theorem. Here, we state these two theorems without proof. We will use Hall's theorem in section 3.

A *vertex cover* of a graph G is any set C of vertices of G such that every edge of G has at least one endpoint in C. An example of a vertex cover in a graph is given below (vertices of the vertex cover are in red).

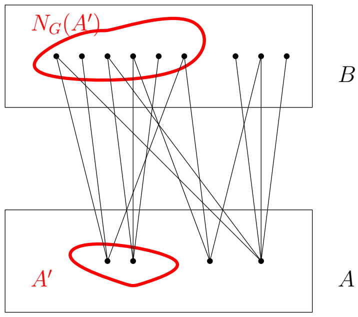


Note that for any graph G, any vertex cover of G is of size at least $\nu(G)$. Indeed, if C is a vertex cover of G, and M is a matching of G, then C contains at least one endpoint of each edge of M; since no two edges of M share an endpoint, it follows that $|C| \geq |M|$. This inequality holds for any vertex cover C and any matching (and in particular, any maximum matching) M of G; so, any vertex cover of G is of size at least $\nu(G)$. For bipartite graphs, we have the following theorem.

The Kőnig-Egerváry theorem. *The maximum size of a matching in a bipartite graph is equal to the minimum size of a vertex cover in that graph.*

For a graph G and a set $X \subseteq V(G)$, we denote by $N_G(X)$ the set of all vertices in $V(G) \setminus X$ that have at least one neighbor in X, i.e. $N_G(X) := \{y \in V(G) \setminus X \mid \exists x \in X \text{ s.t. } xy \in E(G)\}$.

[2]However, there are a great many graphs on an even number of vertices that have no perfect matching. Edgeless graphs are an obvious example, but there are many others.



Hall's theorem. *Let G be a bipartite graph with bipartition (A, B). Then the following are equivalent:*

(a) all sets $A' \subseteq A$ satisfy $|A'| \leq |N_G(A')|$;

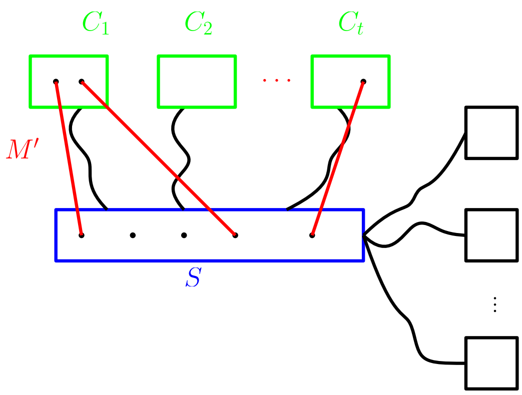
(b) G has an A-saturating matching.

3 The Gallai-Edmonds decomposition

An *odd component* of a graph G is a (connected) component of G that has an odd number of vertices. We denote by $\text{odd}(G)$ the number of odd components of G. In section 4, we will give a formula linking the matching number $\nu(G)$ with the number of odd components of induced subgraphs of G (see the Tutte-Berge formula in section 4). In this section, we develop the technical tools needed to prove the formula.

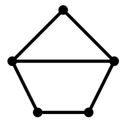
Remark 3.1. *Let G be a graph. Then for all $S \subseteq V(G)$, we have that $\nu(G) \leq \frac{|V(G)|+|S|-odd(G \setminus S)}{2}$.*

Proof. Fix $S \subseteq V(G)$, set $t := \text{odd}(G \setminus S)$, and let $C_1, \dots, C_t$ be the odd components of $G \setminus S$.

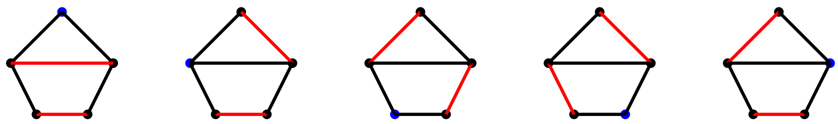


Fix any matching M in G. Let M' be the set of all edges of M that have one endpoint in S and the other one in $V(C_1) \cup \dots V(C_t)$; obviously, $|M'| \leq |S|$. Next, since the components $C_1, \dots, C_t$ are all odd, it follows that at least $t - |M'| \geq t - |S|$ of the components $C_1, \dots, C_t$ have a vertex that is unsaturated by M.[3] So, the total number of vertices of G that are saturated by M is at most $|V(G)| - (t - |S|) = |V(G)| + |S| - t$, and it follows that $|M| \leq \frac{|V(G)|+|S|-t}{2}$. Since the matching M was chosen arbitrarily, we deduce that $\nu(G) \leq \frac{|V(G)|+|S|-t}{2} = \frac{|V(G)|+|S|-\text{odd}(G \setminus S)}{2}$. $\square$

A graph G is *hypomatchable* if it does not have a perfect matching, but for all $v \in V(G)$, the graph $G \setminus v$ does have a perfect matching. Obviously, every hypomatchable graph has an odd number of vertices.[4] For example, the graph below is hypomatchable.



Indeed, deleting any one vertex from the graph above yields a graph that has a perfect matching (as shown below; the vertex that we delete is in blue, and the matching is in red).

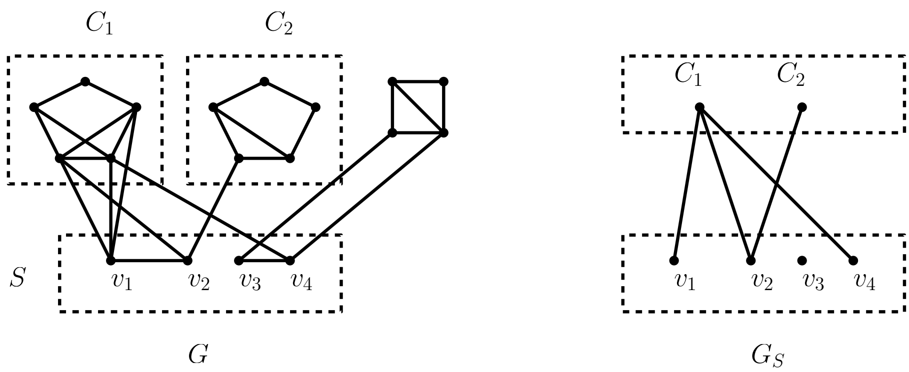


A *hypomatchable component* of a graph G is a component of G that is a hypomatchable graph. Obviously, every hypomatchable component of G is odd.

For a graph G and a set $S \subseteq V(G)$, let us denote by G_S the bipartite graph whose one side of the bipartition is S, and whose other side of the bipartition is the collection of all odd components of $G \setminus S$, and in which a vertex $v \in S$ and an odd component C of $G \setminus S$ are adjacent if and only if v has a neighbor in $V(C)$ in G. An example is shown below.

[3]This is because for all odd components C_i, the number of edges of M that have both endpoints in C_i is at most $\left\lfloor \frac{V(C_i)}{2} \right\rfloor = \frac{|V(C_i)|-1}{2}$; if all vertices of C_i are saturated by M, then there must be an edge of M between S and $V(C_i)$. The number of indices i for which such an edge exists is at most $|M'| \leq |S|$. So, at least $t - |S|$ components C_i have a vertex that is unsaturated by M.

[4]But not all graphs with an odd number of vertices are hypomatchable!



A *Gallai-Edmonds set* in a graph G is a set $S \subseteq V(G)$ that satisfies the following two properties:

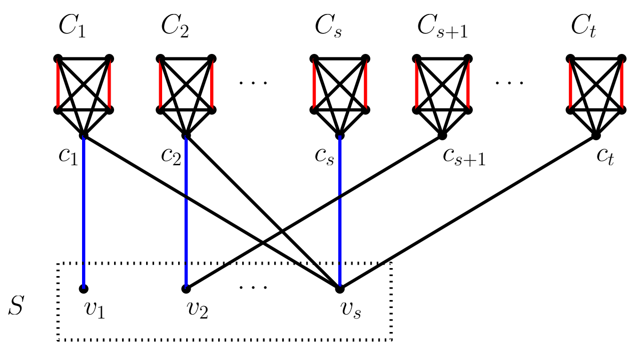
- every component of $G \setminus S$ is hypomatchable (and therefore odd);
- the bipartite graph G_S has an S-saturating matching.

Lemma 3.2. *If S is a Gallai-Edmonds set of a graph G, then*

$$\nu(G) \quad = \quad \frac{|V(G)|+|S|-odd(G\setminus S)}{2}$$

Proof. Let S be a Gallai-Edmonds set of a graph G. By Remark 3.1, we have that $\nu(G) \leq \frac{|V(G)|+|S|-\mathrm{odd}(G\setminus S)}{2}$. It remains to show that $\nu(G) \geq \frac{|V(G)|+|S|-\mathrm{odd}(G\setminus S)}{2}$. To simplify notation, set $n := |V(G)|$, $s := |S|$, and $t := \mathrm{odd}(G \setminus S)$. We must show that $\nu(G) \geq \frac{n+s-t}{2}$. We will prove this by exhibiting a matching M in G of size $\frac{n+s-t}{2}$.

Let $C_1, \dots, C_t$ be the odd components of $G \setminus S$ (since all components of $G \setminus S$ are hypomatchable and therefore odd, we see that $C_1, \dots, C_t$ are in fact all the components of $G \setminus S$), and set $S = \{v_1, \dots, v_s\}$.



Since S is a Gallai-Edmonds set, we know that G_S has an S-saturating matching, call it M_S. By symmetry, we may assume that $M_S = \{v_1C_1, \dots, v_sC_s\}$. For each $i \in \{1, \dots, s\}$, choose a vertex $c_i \in V(C_i)$ such that $v_ic_i \in E(G)$.[5] For all $i \in \{s+1, \dots, t\}$, choose any vertex $c_i \in C_i$. Further, since S is a

[5]Such a vertex c_i must exist because v_i and C_i are adjacent in G_S.

Gallai-Edmonds set, we know that for all $i \in \{1, \dots, t\}$, C_i is hypomatchable, and in particular, $C_i \setminus c_i$ has a perfect matching, call it M_i; clearly, $|M_i| = \frac{|V(C_i)|-1}{2}$. Now, set $M := \{v_1 c_1, \dots, v_s c_s\} \cup M_1 \cup \dots \cup M_t$. Then M is a matching in G. Moreover, M saturates all but $t - s$ vertices of G (indeed, the only vertices of G unsaturated by M are $c_{s+1}, \dots, c_t$), and so $|M| = \frac{n-(t-s)}{2} = \frac{n+s-t}{2}$. □

Lemma 3.3. *Every graph has a Gallai-Edmonds set.*

Proof. Let G be a graph, and assume inductively that every graph on fewer than $|V(G)|$ vertices has a Gallai-Edmonds set.

Choose a set $S \subseteq V(G)$ so that $\mathrm{odd}(G \setminus S) - |S|$ is as large as possible, and subject to that, $|S|$ is as large as possible. Our goal is to show that S is a Gallai-Edmonds set.

Claim 1. All components of $G \setminus S$ are odd.

Proof of Claim 1. Suppose otherwise, and fix a component C of $G \setminus S$ that has an even number of vertices. Fix $v \in V(C)$, and set $S' := S \cup \{v\}$. Since $|V(C)|$ is even, we see that the odd components are precisely the odd components of $G \setminus S$, plus the odd components of $C \setminus v$. Furthermore, since $|V(C)|$ is even, we see that $|V(C) \setminus \{v\}|$ is odd, and so $C \setminus v$ has at least one odd component. Thus,

$$\mathrm{odd}(G \setminus S') \quad = \quad \mathrm{odd}(G \setminus S) + \mathrm{odd}(C \setminus v) \quad \geq \quad \mathrm{odd}(G \setminus S) + 1,$$

and consequently (since $|S'| = |S| + 1$), we have that

$$\begin{aligned} \mathrm{odd}(G \setminus S') - |S'| \quad &\geq \quad \Big(\mathrm{odd}(G \setminus S) + 1\Big) - (|S| + 1) \\ &= \quad \mathrm{odd}(G \setminus S) - |S|. \end{aligned}$$

Since $|S'| > |S|$, this contradicts the choice of S. This proves Claim 1. ■

Claim 2. All components of $G \setminus S$ are hypomatchable.

Proof of Claim 2. Suppose otherwise, and fix a component C of $G \setminus S$ and a vertex $v \in V(C)$ such that $C \setminus v$ does not have a perfect matching. By Claim 1, $C \setminus v$ has an even number of vertices; since $C \setminus v$ does not have a perfect matching, it follows that $\nu(C \setminus v) \leq \frac{|V(C) \setminus \{v\}|}{2} - 1 = \frac{|V(C)|-3}{2}$. By the induction hypothesis, $C \setminus v$ has a Gallai-Edmonds set, call it S_C. Thus,

$$\begin{aligned} \tfrac{|V(C)|-3}{2} \quad &\geq \quad \nu(C \setminus v) \\ &= \quad \frac{|V(C \setminus v)| + |S_C| - \mathrm{odd}\Big((C \setminus v) \setminus S_C\Big)}{2} \qquad \text{by Lemma 3.2} \\ &= \quad \frac{|V(C)| - 1 + |S_C| - \mathrm{odd}\Big((C \setminus v) \setminus S_C\Big)}{2}, \end{aligned}$$

and consequently,

$$\mathrm{odd}\Big((C \setminus v) \setminus S_C\Big) \;\; \geq \;\; |S_C| + 2.$$

Now, set $S' := S \cup \{v\} \cup S_C$. Clearly, the odd components of $G \setminus S'$ are precisely the odd components of $G \setminus S$ other than C, plus the odd components of $(C \setminus v) \setminus S_C$, and so

$$\begin{aligned}
\mathrm{odd}(G \setminus S') &= \mathrm{odd}(G \setminus S) - 1 + \mathrm{odd}\Big((C \setminus v) \setminus S_C\Big) \\
&\geq \mathrm{odd}(G \setminus S) - 1 + (|S_C| + 2) \\
&= \mathrm{odd}(G \setminus S) + |S_C| + 1 \\
&= \mathrm{odd}(G \setminus S) + (|S'| - |S|),
\end{aligned}$$

and we deduce that

$$\mathrm{odd}(G \setminus S') - |S'| \;\; \geq \;\; \mathrm{odd}(G \setminus S) - |S|.$$

Since we also have that $|S'| > |S|$, this contradicts the choice of S. This proves Claim 2. ■

Claim 3. G_S has an S-saturating matching.

Proof of Claim 3. Suppose otherwise. Then by Hall's theorem, there exists a set $X \subseteq S$ such that $|X| > |N_{G_S}(X)|$. Set $S' := S \setminus X$. Then all odd components of $G \setminus S$ other than the ones in $N_{G_S}(X)$ are still odd components of $G \setminus S'$, and we compute:

$$\begin{aligned}
\mathrm{odd}(G \setminus S') &\geq \mathrm{odd}(G \setminus S) - |N_{G_S}(X)| \\
&> \mathrm{odd}(G \setminus S) - |X| \\
&= \mathrm{odd}(G \setminus S) - (|S| - |S'|) \\
&= \mathrm{odd}(G \setminus S) - |S| + |S'|,
\end{aligned}$$

and it follows that

$$\mathrm{odd}(G \setminus S') - |S'| \;\; > \;\; \mathrm{odd}(G \setminus S) - |S|,$$

contrary to the choice of S. This proves Claim 3. ■

By Claims 2 and 3, we have that S is a Gallai-Edmonds set of G. □

4 The Tutte-Berge formula and Tutte's theorem

The Tutte-Berge formula. *Every graph G satisfies*

$$\nu(G) \quad = \quad \tfrac{1}{2} \min_{U \subseteq V(G)} \Big(|V(G)| + |U| - odd(G \setminus U) \Big).$$

Proof. Fix a graph G. By Lemma 3.3, G contains a Gallai-Edmonds set, call it S. Then

$$\begin{aligned} \nu(G) \quad &= \quad \tfrac{|V(G)|+|S|-\mathrm{odd}(G \setminus S)}{2} && \text{by Lemma 3.2} \\ &\geq \quad \tfrac{1}{2} \min_{U \subseteq V(G)} \Big(|V(G)| + |U| - \mathrm{odd}(G \setminus U) \Big). \end{aligned}$$

The reverse inequality follows immediately from Remark 3.1. □

Tutte's theorem. *A graph G has a perfect matching if and only if every set $S \subseteq V(G)$ satisfies $|S| \geq odd(G \setminus S)$.*

Proof. Fix a graph G. Clearly, the following are equivalent:

(a) every set $S \subseteq V(G)$ satisfies $|S| \geq \mathrm{odd}(G \setminus S)$;

(b) $\min_{U \subseteq V(G)} \Big(|V(G)| + |U| - \mathrm{odd}(G \setminus U) \Big) \geq |V(G)|$.

By the Tutte-Berge formula, (b) is equivalent to

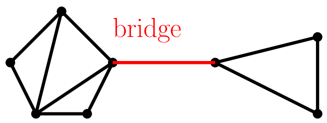
(c) $\nu(G) \geq \frac{|V(G)|}{2}$.

But clearly, (c) holds if and only if G has a perfect matching.[6] So, (a) holds if and only if G has a perfect matching, which is what we needed to show. □

5 Petersen's theorem

For a nonnegative integer k, a graph G is *k-regular* if all vertices of G are of degree k. A graph is *cubic* if it is 3-regular.

A *bridge* in a graph G is an edge $e \in E(G)$ such that $G - e$ has more components than G. A graph is *bridgeless* if it has no bridge.



[6]Indeed, every graph G satisfies $\nu(G) \leq \frac{|V(G)|}{2}$. So, (c) is in fact equivalent to $\nu(G) = \frac{|V(G)|}{2}$. But $\nu(G) = \frac{|V(G)|}{2}$ if and only if G has a perfect matching.

Petersen's theorem. *Every cubic, bridgeless graph has a perfect matching.*[7]

Proof. Fix a cubic, bridgeless graph G. We will apply Tutte's theorem. Fix $S \subseteq V(G)$; we must show that $|S| \geq \text{odd}(G \setminus S)$.

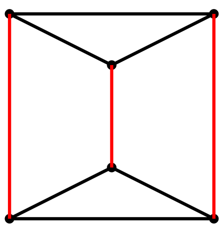
> **Claim.** For all odd components C of $G \setminus S$, there are at least three edges between S and $V(C)$ in G.

Proof of the Claim. Suppose that C is an odd component of $G \setminus S$, and let ℓ be the number of edges between S and $V(C)$. Since G is cubic, we have that $\sum_{v \in V(C)} d_G(v) = 3|V(C)|$;[8] since C is an odd component, we see that $3|V(C)|$ is odd, and consequently, $\sum_{v \in V(C)} d_G(v)$ is odd. On the other hand, every edge incident with a vertex in $V(C)$ either has both its endpoints in $V(C)$, or has one endpoint in $V(C)$ and the other one in S; so, $\sum_{v \in V(C)} d_G(v) = 2|E(G[C])| + \ell$. Since $\sum_{v \in V(C)} d_G(v)$ is odd, we see that ℓ is odd. If $\ell = 1$, then the unique edge between S and $V(C)$ is a bridge in G, contrary to the fact that G is bridgeless. So, $\ell \geq 3$. This proves the Claim. ■

Set $t := \text{odd}(G \setminus S)$. By the Claim, the number of edges between S and $V(G) \setminus S$ is at least $3t$. On the other hand, since G is cubic, the total number of edges incident with at least one vertex of S as at most $3|S|$.[9] Thus, $3t \leq 3|S|$, i.e. $|S| \geq t = \text{odd}(G \setminus S)$. Since $S \subseteq V(G)$ was chosen arbitrarily, Tutte's theorem guarantees that G has a perfect matching. □

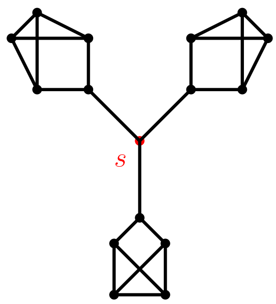
The bridgelessness requirement from Petersen's theorem is necessary, as the example below shows.

[7] Here is an example of a cubic, bridgeless graph, with a perfect matching shown in red.



[8] As usual, $d_G(v)$ is the degree of v in G.

[9] Note that we are double counting edges whose both endpoints are in S. Hence, the number of edges incident with at least one vertex of S is at most $3|S|$, and not necessarily exactly $3|S|$.



The graph above (call it G) is cubic, but not bridgeless. If we set $S := \{s\}$, then $G \setminus S$ has three odd components, and so $|S| < \text{odd}(G \setminus S)$. Thus, by Tutte's theorem, G does not have a perfect matching.