NDMI012: Combinatorics and Graph Theory 2
HW#9

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due Thursday, May 5, 2022, 15:40 (at the beginning of the tutorial)

Remark: Bring your HW to the beginning of the tutorial. If you must miss the tutorial, please e-mail your HW to me (ipenev@iuuk.mff.cuni.cz) as a **PDF attachment** (no other format will be accepted).

Problem 1 (20 points)**.** *Let G be a graph such that $\delta(G) \geq 2$.*[1] *Prove that G contains a cycle of length at least $\delta(G) + 1$.*

> ***Hint:*** *Consider a path P in G of maximum length, at show that a subpath of P, plus an additional edge of G, form the needed cycle.*
>
> ***Remark:*** *This problem should be useful in one of the subsequent two problems (figure out where!).*

Definition. *For an integer $k \geq 0$, a graph G is k-*vertex-connected *(or simply k-*connected*) if the following two conditions are satisfied:*

- *$|V(G)| \geq k + 1$;*
- *for all $S \subseteq V(G)$ such that $|S| \leq k - 1$, the graph $G \setminus S$ is connected.*

The vertex-connectivity *(or simply* connectivity*) of G, denoted by $\kappa(G)$, is the largest integer $k \geq 0$ such that G is k-connected.*

Problem 2 (50 points)**.** *Let G be a graph such that $G \not\cong K_2$ (i.e. G is not a complete graph on two vertices) and $\kappa(G) \geq \alpha(G)$.*[2]

(a) [10 points] Prove that G has a cycle of length at least $\kappa(G) + 1$.

[1] $\delta(G) := \min\{d_G(v) \mid v \in V(G)\}$
[2] $\alpha(G) := \max\{|S| \mid S \text{ is a stable set of } G\}$

(b) *[10 points] Let C be a cycle of G of maximum length (by part (a), C exists). Prove that, if $V(C) \subsetneq V(G)$, then for every component H of $G \setminus V(C)$, at least $\kappa(G)$ many vertices of C have a neighbor in H.*

(c) *[30 points] Prove that G is Hamiltonian.*

> ***Hint:*** *Let C be a cycle of G of maximum length; if $V(C) = V(G)$, then C is a Hamiltonian cycle of G. Otherwise, use part (b) to obtain either a cycle that is "too long" or a stable set that is "too big."*

Problem 3 (30 points)**.** *Prove that any non-Hamiltonian graph on n vertices has at most $\binom{n-1}{2} + 1$ edges.*