NDMI012: Combinatorics and Graph Theory 2
HW#8

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due Thursday, April 21, 2022, 15:40 (at the beginning of the tutorial)

Remark: Bring your HW to the beginning of the tutorial. If you must miss the tutorial, please e-mail your HW to me (ipenev@iuuk.mff.cuni.cz) as a **PDF attachment** (no other format will be accepted).

The Kőnig-Egerváry theorem. *The maximum size of a matching in a bipartite graph is equal to the minimum size of a vertex cover in that graph.*

Problem 1 (40 points)**.** *Use Dilworth's theorem to prove the Kőnig-Egerváry theorem (stated above).*[1]

> ***Hint:*** *What is the most obvious way to "transform" a bipartite graph into a partially ordered set?*

Definition. *Given graphs H and K on disjoint vertex sets, and given a vertex $u \in V(H)$, we say that a graph G is obtained by* substituting K *for* u *in* H *provided the following hold:*

- $V(G) = (V(H) \setminus \{u\}) \cup V(K)$;
- $G[V(H) \setminus \{u\}] = H \setminus u$;
- $G[V(K)] = K$;
- *for all* $u' \in V(K)$ *and* $v \in V(H) \setminus \{u\}$, u' *is adjacent to* v *in* G *if and only if* u *is adjacent to* v *in* H.

[1] The Kőnig-Egerváry theorem was proven in Combinatorics & Graphs 1. Here, you are asked to give a different proof.

Problem 2 (60 points). *In this problem, you may use* ***neither*** *the Perfect Graph Theorem* ***nor*** *the Strong Perfect Graph Theorem.*

(a) *[30 points] Prove that the graph obtained by substituting a complete graph for a vertex of a perfect graph is perfect.*[2]

> ***Hint:*** *Imitate the proof of the fact that α-perfection is preserved under vertex duplication. Think of a proper coloring as a partition into stable sets.*

(b) *[30 points] Prove that the graph obtained by substituting a perfect graph for a vertex of a perfect graph is perfect.*[3]

> ***Hint:*** *Suppose G is obtained by substituting K for a vertex u of H, where K and H are perfect graphs on disjoint vertex sets. Start by substituting a complete graph on $\omega(K)$ vertices for u in H, and use part (a). Now what?*

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[2] In other words, prove that if H and K are graphs on disjoint vertex sets, H is perfect, K is complete, and $u \in V(H)$, then the graph G obtained by substituting K for u in H is perfect.

[3] In other words, prove that if H and K are perfect graphs on disjoint vertex sets and $u \in V(H)$, then the graph G obtained by substituting K for u in H is perfect.