NDMI012: Combinatorics and Graph Theory 2
HW#6

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due Thursday, April 7, 2022, 15:40 (at the beginning of the tutorial)

Remark: Bring your HW to the beginning of the tutorial. If you must miss the tutorial, please e-mail your HW to me (ipenev@iuuk.mff.cuni.cz) as a **PDF attachment** (no other format will be accepted).

Problem 1 (30 points)**.** *Let k be a positive integer, and let G be a graph on $k+1$ vertices and of chromatic number k. Prove that $\omega(G) = k$.*

> ***Hint:*** *It may be useful to think of a coloring as a partition into stable sets (color classes).*

Problem 2 (30 points)**.** *Prove that every graph G satisfies*

$$\chi(G) + \chi(\overline{G}) \quad \leq \quad |V(G)| + 1,$$

where $\overline{G}$ denotes the complement of G.

> ***Hint:*** *Use induction on $|V(G)|$.*

Problem 3 (40 points)**.** *A* graph-stable pair *is an ordered pair $(G, \mathscr{S})$, where G is a graph, and $\mathscr{S}$ is some set of stable sets of G.*[1] *We recursively construct a sequence $\{(G_k, \mathscr{S}_k)\}_{k=1}^{\infty}$ of graph-stable pairs as follows.*

> *Let $G_1 = K_1$ and $\mathscr{S}_1 = \{V(G_1)\}$. Next, let k be a positive integer, and assume that the graph-stable pair $(G_k, \mathscr{S}_k)$ has been constructed. We construct $(G_{k+1}, \mathscr{S}_{k+1})$ as follows. For each stable set $S \in \mathscr{S}_k$, let $(H_S, \mathscr{S}(H_S))$ be a copy of $(G_k, \mathscr{S}_k)$. We form G_{k+1} by first taking the disjoint union of G_k and the graphs*

[1] This means that every member of $\mathscr{S}$ is a stable set of G. Note that $\mathscr{S}$ need not contain **all** stable sets of G (but only some).

H_S ($S \in \mathscr{S}_k$),[2] *and then for each $S \in \mathscr{S}_k$ and $T \in \mathscr{S}(H_S)$, adding a new vertex $v_{S,T}$ whose neighborhood in G_{k+1} is precisely the set T. Finally, set $\mathscr{S}_{k+1} = \{S \cup T \mid S \in \mathscr{S}_k, T \in \mathscr{S}(H_S)\} \cup \{S \cup \{v_{S,T}\} \mid S \in \mathscr{S}_k, T \in \mathscr{S}(H_S)\}$.*

(a) [10 points] Prove that for all positive integers k, G_k is triangle-free.

(b) [20 points] Prove that for all positive integers k, and all proper (not necessarily optimal) colorings ϕ of G_k, there exists some $S \in \mathscr{S}_k$ such that ϕ uses at least k colors on S.

(c) [10 points] Prove that for all positive integers k, $\chi(G_k) = k$. You may use the statement of (b) even if you did not prove it.

[2] Now we have $|\mathscr{S}_k| + 1$ disjoint copies of G_k; by construction, there are no edges between any two of them.