NDMI012: Combinatorics and Graph Theory 2
HW#3

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due Thursday, March 17, 2022, 15:40 (at the beginning of the tutorial)

Remark: Bring your HW to the beginning of the tutorial. If you must miss the tutorial, please e-mail your HW to me (ipenev@iuuk.mff.cuni.cz) as a **PDF attachment** (no other format will be accepted).

Problem 1 (40 points)**.** *Let $n \geq 5$ be an integer. Prove that in any drawing of K_n in the plane,*[1] *there are at least $\frac{1}{5}\binom{n}{4}$ many edge crossings (i.e. at least $\frac{1}{5}\binom{n}{4}$ many pairs of edges cross).*

> ***Hint:*** *Use the fact that K_5 is non-planar, and so in any drawing of K_5 in the plane, there is at least one edge crossing. How many copies of K_5 can you find in K_n? And then what?*

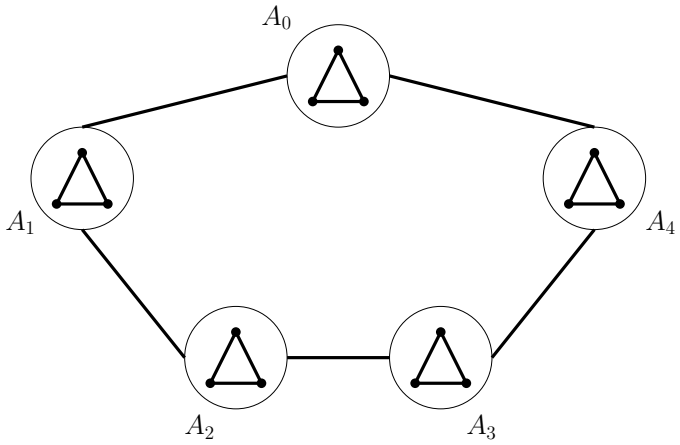
Problem 2 (60 points)**.** *Let G be a graph whose vertex set can be partitioned into five sets, call them A_0, A_1, A_2, A_3, A_4 (with indices understood to be in $\mathbb{Z}_5$), satisfying the following properties:*

- $|A_0| = |A_1| = |A_2| = |A_3| = |A_4| = 3$*;*
- *A_0, A_1, A_2, A_3, A_4 are all cliques;*
- *for all $i \in \mathbb{Z}_5$, A_i is complete to $A_{i-1} \cup A_{i+1}$ and anticomplete to $A_{i-2} \cup A_{i+2}$.*[2]

Schematically, the graph G can be represented as in the picture below (a line between two circles indicates that all vertices inside one of the circles are adjacent to all vertices inside the other circle).

[1] We are assuming that distinct vertices are assigned distinct points in the plane, and that no vertex is drawn in the interior of an edge. However, edges may cross.

[2] This means that every vertex of A_i is adjacent to every vertex of $A_{i-1} \cup A_{i+1}$, and there are no edges between A_i and $A_{i-2} \cup A_{i+2}$.



(a) [20 points] Prove that $\chi(G) = 8$.

(b) [20 points] Prove that $K_8 \not\preceq_t G$.

(c) [20 points] Prove that $K_8 \preceq_m G$.

Remark: *Note that (a) and (b) imply that* G *is a counterexample to Hajós' conjecture. However, by (a) and (c),* G *is* ***not*** *a counterexample to Hadwiger's conjecture.*