NDMI012: Combinatorics and Graph Theory 2

Lecture #11
Burnside's lemma and applications

Irena Penev

1 Groups

A *group* is a set G, together with a binary operation $\circ$, satisfying the following properties:

- $\circ$ is associative, i.e. for all $g_1, g_2, g_3 \in G$, $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$;
- there exists some $e \in G$, called the *identity element*, such that for all $g \in G$, $e \circ g = g \circ e = g$;
- for all $g \in G$, there exists some $g' \in G$, called the *inverse* of g, such that $g \circ g' = g' \circ g = e$.

Usually, for $g_1, g_2 \in G$, we write "$g_1 g_2$" instead of "$g_1 \circ g_2$." It is easy to show that the identity element is unique;[1] typically, this identity element is denoted by 1_G, or simply 1. Furthermore, it can be shown that each element of G has a unique inverse;[2] the unique inverse of an element $g \in G$ is usually denoted by g^{-1}.

For a set X, $\mathrm{Sym}(X)$ is the group of all permutations of X;[3] the group

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[1]Indeed, suppose e_1, e_2 are identity elements of G. Then $e_1 e_2 = e_1$ (because e_2 is an identity element), and $e_1 e_2 = e_2$ (because e_1 is an identity element). So, $e_1 = e_2$.

[2]Indeed, fix $g \in G$, and suppose that $g_1, g_2 \in G$ are inverses of g. Then

$$\begin{aligned} g_1 &= g_1 1_G && \text{because } 1_G \text{ is the identity element} \\ &= g_1(g g_2) && \text{because } g_2 \text{ is an inverse of } g \\ &= (g_1 g) g_2 && \text{because } \circ \text{ is associative} \\ &= 1_G g_2 && \text{because } g_1 \text{ is an inverse of } g \\ &= g_2 && \text{because } 1_G \text{ is an identity element} \end{aligned}$$

which is what we needed.

[3]A *permutation* of X is a bijection between X and itself.

operation is the composition of functions, and the identity element is the identity function on X, denoted by Id_X.[4] For a positive integer n, the group of permutations of the set $\{1, \ldots, n\}$ is denoted by $\mathrm{Sym}(n)$ or Sym_n. A permutation $\pi \in \mathrm{Sym}(n)$ can be denoted by

$$\begin{pmatrix} 1 & 2 & \ldots & n \\ \pi(1) & \pi(2) & \ldots & \pi(n) \end{pmatrix}.$$

Recall that each permutation in $\mathrm{Sym}(n)$ can be represented as a composition of disjoint cycles. For example, the following permutation in $\mathrm{Sym}(5)$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix}$$

can be represented as $(143)(25)$. Cycles of length one are usually omitted (when n is clear from context). For example, in $\mathrm{Sym}(5)$, instead of $(124)(3)(5)$, we typically write simply (124).

2 Group actions and Burnside's lemma

A *left action* (or simply *action*)[5] of a group G on a set X is a function $a : G \times X \to X$ that satisfies the following two properties:

- for all $x \in X$, $a(1_G, x) = x$.
- for all $g_1, g_2 \in G$ and $x \in X$, $a(g_1, a(g_2, x)) = a(g_1 g_2, x)$.

Often, instead of $a(g, x)$, we write simply $g \cdot x$. So, using this notation, the axioms above become:

- for all $x \in X$, $1_G \cdot x = x$.
- for all $g_1, g_2 \in G$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.

Note that these axioms imply that, for all $g \in G$ and $x, y \in X$, if $g \cdot x = y$, then $g^{-1} \cdot y = x$.[6]

Example 2.1. *Any group G acts on itself in a natural way: for all $g \in G$ and $x \in G$,[7] we set $g \cdot x = gx$.*

Given an action $a : G \times X \to X$ of a group G on a set X, and an element $g \in G$, we define a function $a_g : X \to X$ by setting $a_g(x) = a(g, x)$ for all $x \in X$. As our next proposition shows, a_g is simply a permutation of X. So, we can think of group action as a collection of permutations (one permutation of the set X for each member g of the group G), which must satisfy certain additional properties (as in the definition of group action).

[4] That is, $\mathrm{Id}_X : X \to X$ satisfies $\mathrm{Id}_X(x) = x$ for all $x \in X$.

[5] Yes, there is also such a thing as "right action," but we will not consider this here.

[6] Indeed, if $g \cdot x = y$, then $g^{-1} \cdot y = g^{-1} \cdot (g \cdot x) = (g^{-1} g) \cdot x = 1_G \cdot x = x$.

[7] Here, $X = G$.

Proposition 2.2. *Let $a : G \times X \to X$ be an action of a group G on a set X. Then for all $g \in G$, the function a_g is a permutation of X.*

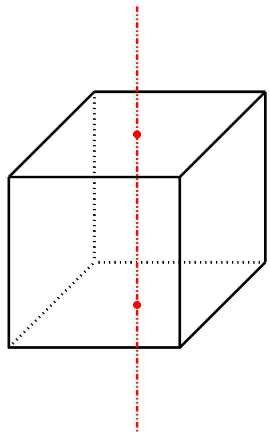
Proof. Fix $g \in G$, and consider its inverse g^{-1}. Then for all $x \in X$, we have that

$$\begin{aligned} a_{g^{-1}} \circ a_g(x) &= a(g^{-1}, a(g,x)) \\ &= a(g^{-1}g, x) \\ &= a(1_G, x) \\ &= x, \end{aligned}$$

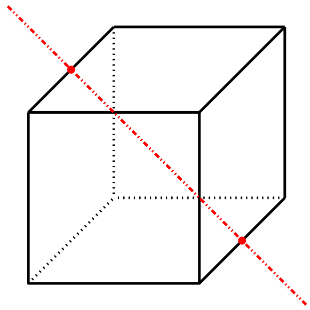
and so $a_{g^{-1}} \circ a_g = \mathrm{Id}_X$. A completely analogous argument shows that $a_g \circ a_{g^{-1}} = \mathrm{Id}_X$. So, $a_g : X \to X$ is a bijection with inverse $a_{g^{-1}}$, and the result follows. □

Example 2.3. *Consider a cube in $\mathbb{R}^3$, and let R_{cube} be the group of rotations of $\mathbb{R}^3$ that map this cube to itself. (Here, the group operation is the composition of functions/rotations, and the identity element is the identity function on $\mathbb{R}^3$.) The group R_{cube} acts on the faces of the cube in a natural way: for each rotation $r \in R_{cube}$ and each face f of the cube, $r \cdot f$ is the face of the cube to which the rotation r maps/moves the face f. We note that $|R_{cube}| = 24$. Indeed, the rotations in R_{cube} are as follows:*

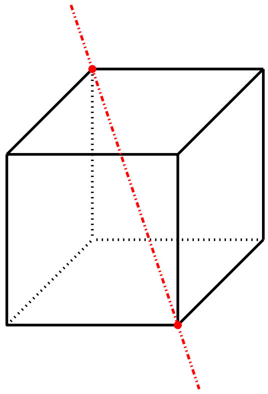
- *the identity function;*
- *nine rotations about an axis passing though centers of opposite faces of the cube (there are three choices of axis, and for each choice, we can rotate by $90°$, $180°$, or $270°$);*



- *six rotations about an axis passing through centers of opposite edges of the cube (there are six choices of axis, and for each choice, we can rotate only by $180°$);*



- *eight rotations around axes passing through opposite vertices of the cube (there are four choices of axis, and in each case, we can rotate by* $120°$ *or* $240°$*).*



We now need a few definitions. Suppose that a is an action of a group G on a set X. Then

- for each $g \in G$, a *fixed point* of g is any $x \in X$ such that $g \cdot x = x$, and we set $X^g := \{x \in X \mid g \cdot x = x\}$;[8]

- for each $x \in X$, we define the *stabilizer* of x to be $G_x := \{g \in G \mid g \cdot x = x\}$;

- for each $x \in X$, we define the *orbit* of x to be $G \cdot x := \{g \cdot x \mid g \in G\}$.[9]

Proposition 2.4. *Let a be an action of a group G on a set X. Then*

- *for all $x \in X$, we have that $x \in G \cdot x$;*

- *the orbits of the action a form a partition of X.*

[8]So, X^g is the set of all fixed points of g (with respect to the action a).
[9]So, the orbit of x is the set of all elements of x that G can "move" x to.

Proof. First, since $1_G \cdot x = x$, we see that $x \in G \cdot x$. In particular, each element of X belongs to some orbit. It remains to show that any two distinct orbits are disjoint. So, fix $x_1, x_2 \in X$; we must show that $G \cdot x_1$ and $G \cdot x_2$ are either equal or disjoint. Suppose that $G \cdot x_1$ and $G \cdot x_2$ are not disjoint; we claim that $G \cdot x_1 = G \cdot x_2$. We will show that $G \cdot x_1 \subseteq G \cdot x_2$; the proof of the reverse inclusion is analogous. Fix some $y \in (G \cdot x_1) \cap (G \cdot x_2)$. Then there exist $g_1, g_2 \in G$ such that $y = g_1 \cdot x_1$ and $y = g_2 \cdot x_2$; so, $g_1 \cdot x_1 = g_2 \cdot x_2$. But then $x_1 = (g_1^{-1} g_2) \cdot x_2$.[10] Now, for all $g \in G$, we have that $g \cdot x_1 = g \cdot ((g_1^{-1} g_2) \cdot x_2) = (g g_1^{-1} g_2) \cdot x_2$, and so $g \cdot x_1 \in G \cdot x_2$. Thus, $G \cdot x_1 \subseteq G \cdot x_2$, and we are done. □

Given an action a of a group G on a set X, we denote by X/G the partition of X into orbits of a. So, $|X/G|$ is the number of orbits of a.

Next, given an action a of a group G on a set X, and given $x, y \in X$, we set $M_a(x, y) := \{g \in G \mid g \cdot x = y\}$. Note that $M_a(x, x) = G_x$, and that $M_a(x, y) \neq \emptyset$ if and only if $y \in G \cdot x$.

Lemma 2.5. *Let a be an action of a finite group G on a finite set X, and let $x \in X$. Then for all $y \in G \cdot x$, we have that $|M_a(x, y)| = |G_x|$.*

Proof. Fix $y \in G \cdot x$, and fix any $g_y \in G$ such that $g_y \cdot y = x$. We now define a function $f : G \to G$ by setting $f(g) = g_y g$; since G is a group, f is one-to-one. Now, our goal is to show that $f[G_x] = M_a(x, y)$; this will prove that $|G_x| = |M_a(x, y)|$, which is what we need.

First, fix $g \in G_x$. Then $f(g) \cdot x = (g_y g) \cdot x = g_y \cdot (g \cdot x) = g_y \cdot x = y$, and so $f(g) \in M_a(x, y)$. Thus, $f[G_x] \subseteq M_a(x, y)$.

On the other hand, fix any $g' \in M_a(x, y)$. Then $(g_y^{-1} g') \cdot x = g_y^{-1} \cdot (g' \cdot x) = g_y^{-1} \cdot y = g_y^{-1} \cdot (g_y \cdot x) = (g_y^{-1} g_y) \cdot x = 1_G \cdot x = x$, and so $g_y^{-1} g' \in G_x$. But now $f(g_y^{-1} g') \cdot x = (g_y g_y^{-1} g') \cdot x = g' \cdot x = y$. Thus, $M_a(x, y) \subseteq f[G_x]$.

We have now shown that $f[G_x] = M_a(x, y)$. Since f is one-to-one, it follows that $|M_a(x, y)| = |G_x|$, which is what we needed. □

As an easy corollary of Lemma 2.5, we get the following theorem.

The orbit-stabilizer theorem. *Let a be an action of a finite group G on a finite set X. Then for all $x \in X$, we have that $|G \cdot x| = \frac{|G|}{|G_x|}$.*

Proof. Fix $x \in X$, and note that sets of the form $M_a(x, y)$, with $y \in G \cdot x$,

[10] Indeed, $x_1 = 1_G \cdot x_1 = (g_1^{-1} g_1) \cdot x_1 = g_1^{-1} \cdot (g_1 \cdot x_1) = g_1^{-1} \cdot (g_2 \cdot x_2) = (g_1^{-1} g_2) \cdot x_2$.

form a partition of G,[11] and so

$$\begin{aligned} |G| &= |\bigcup_{y \in G \cdot x} M_a(x,y)| \\ &= \sum_{y \in G \cdot x} |M_a(x,y)| \\ &= \sum_{y \in G \cdot x} |G_x| \qquad \text{by Lemma 2.5} \\ &= |G \cdot x||G_x|, \end{aligned}$$

and the result follows. □

Lemma 2.6. *Let a be an action of a finite group G on a finite set X. Then*

$$|X/G| = \sum_{x \in X} \frac{1}{|G \cdot x|}.$$

Proof. Set $t := |X/G|$, and let $O_1, \ldots, O_t$ be the orbits of the action a. Then by Proposition 2.4, we have that

- for all $i \in \{1, \ldots, t\}$ and $x \in O_i$, $G \cdot x = O_i$;
- $(O_1, \ldots, O_t)$ is a partition of X.

We now compute

$$\sum_{x \in X} \frac{1}{|G \cdot x|} = \sum_{i=1}^{t} \sum_{x \in O_i} \frac{1}{|G \cdot x|} = \sum_{i=1}^{t} \sum_{x \in O_i} \frac{1}{|O_i|} = t,$$

which is what we needed. □

We are now ready to state and prove Burnside's lemma, which (roughly) states that the number of orbits of an action is equal to the average number of fixed points.

Burnside's lemma. *Let a be an action of a finite group G on a finite set X. Then*

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Proof. Let $F := \{(g,x) \in G \times X \mid g \cdot x = x\}$. We will count $|F|$ in two ways.

On the one hand, for all $g \in G$ and $x \in X$, we have that $(g,x) \in F$ if and only if $x \in X^g$; so,

$$|F| = \sum_{g \in G} |X^g|.$$

[11] That is: for all distinct $y_1, y_2 \in G \cdot x$, we have that $M_a(x,y_1) \cap M_a(x,y_2) = \emptyset$, and $\bigcup_{y \in G \cdot x} M_a(x,y) = G$.

On the other hand, for all $g \in G$ and $x \in X$, we have that $(g, x) \in F$ if and only if $g \in G_x$, and so

$$\begin{aligned} |F| &= \textstyle\sum_{x\in X} |G_x| \\ &= \textstyle\sum_{x\in X} \frac{|G|}{|G\cdot x|} && \text{by the orbit-stabilizer theorem} \\ &= |G| \textstyle\sum_{x\in X} \frac{1}{|G\cdot x|}. \end{aligned}$$

But now

$$|G| \textstyle\sum_{x\in X} \frac{1}{|G\cdot x|} = \textstyle\sum_{g\in G} |X^g|,$$

and consequently,

$$\textstyle\sum_{x\in X} \frac{1}{|G\cdot x|} = \frac{1}{|G|} \textstyle\sum_{g\in G} |X^g|.$$

But by Lemma 2.6, $|X/G| = \sum_{x\in X} \frac{1}{|G\cdot x|}$, and the result follows. □

3 Applications of Burnside's lemma

Example 3.1. *Let R_{cube} be the group of rotations of the cube, as in Example 2.3, and let k be a positive integer. Let B_k be the set of all colorings of the faces of the cube using the color set $\{1, \dots, k\}$. Then R_{cube} acts on the set B_k in the natural way: a rotation $r \in R_{cube}$ maps each element of B_k to an appropriately rotated coloring. Two colorings of the cube are* equivalent *if one can be transformed into the other by a rotation in R_{cube}. Compute the number of non-equivalent colorings of the cube using the color set $\{1, \dots, k\}$.*

Solution. The number of non-equivalent colorings of the cube using the color set $\{1, \dots, k\}$ is precisely equal to the number of orbits of our action of R_{cube} on B_k, which we will compute using Burnside's lemma. We know that $|R_{\text{cube}}| = 24$ (see Example 2.3), and for each $r \in R_{\text{cube}}$, we compute B_k^r as follows.

- If r is the identity rotation, then $|B_k^r| = |B_k| = k^6$.
- If r is a rotation by 90° or 270° about an axis passing though the center of opposite faces (there are a total of six such r's), then r fixes precisely the colorings in which the faces not pierced by the axis have the same color. So, we choose one of k colors for one of the faces pieced by the axis, one of k colors for the other face pierced by the axis, and one of k colors for all the remaining four faces. In total, we get $|B_k^r| = k^3$.
- If r is a rotation by 180° about an axis passing though the center of opposite faces (there are a total of three such r's), then r fixes exactly the the colorings for which the opposite faces that are not pierces by the axis have the same color. There are two pairs of opposite faces not pierced by our axis, and it follows that $|B_k^r| = k^4$.

- If r is a rotation by 180° about an axis passing though the center of opposite edges (there are a total of six such r's), then r fixes exactly the colorings for which the two opposite faces not incident with the edges pierced by the axis have the same color, and in which, for each pierced edge, the two faces incident with this edge have the same color. So, $|B_k^r| = k^3$.

- Finally, if r is a rotation by 120° or 240° about an axis passing though opposite vertices (there are a total of eight such r's), then r fixes exactly the colorings for which the three incident faces with each of the pierces vertices have the same color. So, $|B_k^r| = k^2$.

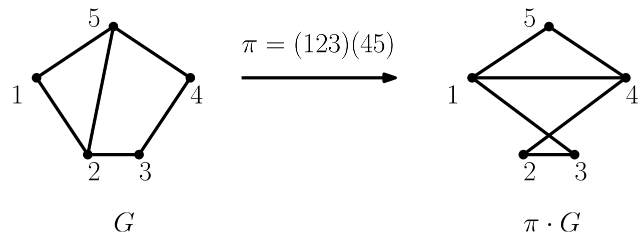
So, by Burnside's lemma, the total number of orbits of our action (and therefore, the total number of non-equivalent colorings) is

$$\frac{1}{|R_{\text{cube}}|}\sum_{r \in R_{\text{cube}}} |B_k^r| \quad = \quad \frac{k^6+6k^3+3k^4+6k^3+8k^2}{24} \quad = \quad \frac{k^6+3k^4+12k^3+8k^2}{24}$$

□

Example 3.2. *Find the number of non-isomorphic graphs on five vertices.*

Solution. Let X be the set of all graphs on the vertex set $\{1, \dots, 5\}$. We let Sym(5) act on X in the natural way: given a graph $G \in X$ and a permutation $\pi \in \text{Sym}(5)$, we let $\pi \cdot G$ be the graph with vertex set $\{1, \dots, 5\}$, in which distinct vertices $i, j \in \{1, \dots, 5\}$ are adjacent if and only if $\pi^{-1}(i)$ and $\pi^{-1}(j)$ are adjacent in G. An example is shown below.



Clearly, two graphs in X are isomorphic if and only if they belong to the same orbit of this action. So, the number of non-isomorphic graphs on five vertices is equal to the number of orbits of our action. We will compute the number of orbits using Burnside's lemma.

Clearly, $|\text{Sym}(5)| = 5!$. We compute the number of fixed points of a permutation $\pi \in \text{Sym}(5)$ according to the cycle structure of π.

- If π is the identity function, then π fixes all elements of X, i.e. $|X^\pi| = |X| = 2^{\binom{5}{2}} = 2^{10}$.

- If $\pi = (ab)$, for distinct $a, b \in \{1, \dots, 5\}$ (note: there are $\binom{5}{2} = 10$ such π's), then π fixes precisely the graphs $G \in X$ such that $N_G(a) \setminus \{b\} = N_G(b) \setminus \{a\}$. So, we can freely select the neighbors of a (the neighbors of b are then forced), and we can choose adjacency between vertices in $\{1,\dots,5\}\setminus\{a,b\}$ arbitrarily. There are 2^4 ways to choose the neighbors of a, and there are $2^{\binom{3}{2}} = 2^3$ ways to choose adjacency between vertices in $\{1,\dots,5\}\setminus\{a,b\}$. So, $|X^\pi| = 2^4 \cdot 2^3 = 2^7$.

- If $\pi = (ab)(cd)$ for pairwise distinct $a,b,c,d \in \{1,\dots,5\}$ (note: there are 15 such π's), then π fixes precisely the graphs $G \in X$ satisfying the following three properties:

  – ac is an edge if and only if bd is an edge,

  – ad is an edge if and only if bc is an edge,

  – the fifth vertex of G (i.e. the unique vertex in $\{1,\dots,5\}\setminus\{a,b,c,d\}$) is adjacent to a if and only if it is adjacent to b, and is adjacent to c if and only if it is adjacent to d.

  So, $|X^\pi| = 2^4$.

- If $\pi = (abc)$, for pairwise distinct $a,b,c \in \{1,\dots,5\}$ (note: there are 20 such π's), then π fixes precisely the graphs $G \in X$ in which $\{a,b,c\}$ is either a clique or a stable set, and each of the remaining two vertices (i.e. vertices in $\{1,\dots,5\}\setminus\{a,b,c\}$) is either complete or anticomplete to $\{a,b,c\}$. So, $|X^\pi| = 2^4$.

- If $\pi = (abc)(de)$, for pairwise distinct $a,b,c,d,e \in \{1,\dots,5\}$ (note: there are 20 such π's), then π fixes precisely the graphs $G \in X$ in which $\{a,b,c\}$ is either a clique or a stable set, and $\{a,b,c\}$ is either complete or anticomplete to $\{d,e\}$. So, $|X^\pi| = 2^3$.

- If $\pi = (a,b,c,d)$, for pairwise distinct $a,b,c,d \in \{1,\dots,5\}$ (note: there are 30 such π's), then π fixes precisely the graphs $G \in X$ in which all the following hold:

  – ab, bc, cd, da are either all edges or all non-edges,

  – ac and bd are either both edges or both non-edges,

  – the fifth vertex of G (i.e. the unique vertex in $\{1,\dots,5\}\setminus\{a,b,c,d\}$) is either complete or anticomplete to $\{a,b,c,d\}$.

  So, $|X^\pi| = 2^3$.

- If $\pi = (abcde)$, for pairwise distinct a,b,c,d,e (note: there are 24 such π's), then π fixes precisely the graphs $G \in X$ in which both the following hold:

  – ab, bc, cd, de, ea are either all edges or all non-edges,

  – ac, bd, ce, da, eb are either all edges or all non-edges.

So, $|X^\pi| = 2^2$.

Now, by Burnside's lemma, we see that the number of orbits of our action is

$$\begin{aligned} |X/\mathrm{Sym}(5)| &= \tfrac{1}{|\mathrm{Sym}(5)|} \sum_{\pi \in \mathrm{Sym}(5)} |X^\pi| \\ &= \tfrac{1}{5!}\left(2^{10} + 10 \cdot 2^7 + 15 \cdot 2^6 + 20 \cdot 2^4 + 20 \cdot 2^3 + 30 \cdot 2^3 + 24 \cdot 2^2\right) \\ &= 34. \end{aligned}$$

So, there are 34 non-isomorphic graphs on five vertices. □