NDMI012: Combinatorics and Graph Theory 2

Lecture #9

The Tutte polynomial

Irena Penev

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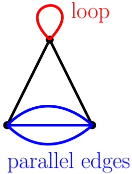
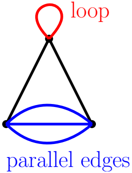
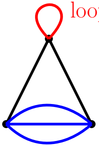
Definition

A *multigraph* is an ordered pair $G = (V(G), E(G))$ s.t. $V(G)$ and $E(G)$ are finite sets (called the *vertex set* and *edge set*, respectively), and each edge (i.e. element of $E(G)$) is associated with two (possibly identical) vertices (i.e. elements of $V(G)$), called its *endpoints*. If an edge has only one endpoint (i.e. its two endpoints are the same), then this edge is called a *loop*. If two distinct edges have the same endpoints, then those edges are *parallel*.





- A *proper (vertex) coloring* of a loopless multigraph G is an assignment of colors to the vertices of G in such a way that, whenever two distinct vertices are joined by an edge (i.e. are the endpoints of the same edge), they receive different colors.



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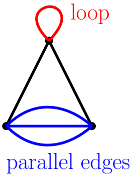

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- If a multigraph has a loop, then it has no proper colorings.
- A *k-coloring* of a multigraph G is a proper coloring of G that uses colors $1, \dots, k$ (not all of these colors need be used).

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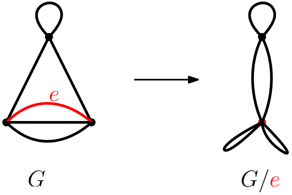
- If e is a non-loop edge of a multigraph G, then the multigraph G/e obtained by *contracting* e is the multigraph obtained by first deleting e, and then identifying its endpoints to a single vertex.



  - Note that edges parallel to e become loops, and it is also possible that new parallel edges are created.

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- There are a number of such polynomials.
- Here, we consider two: the chromatic polynomial and the Tutte polynomial.

Lemma 2.1

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- If G is a loopless multigraph, then $\chi(G)$ is equal to the smallest non-negative integer k s.t. $\pi_G(k) \neq 0$.

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- The *chromatic polynomial* of a multigraph G is the polynomial π_G from the statement of Lemma 2.1.
- If G is a loopless multigraph, then $\chi(G)$ is equal to the smallest non-negative integer k s.t. $\pi_G(k) \neq 0$.
- Note that this implies that computing the chromatic polynomial is NP-hard.
- However, in some special cases, the chromatic polynomial is easy to compute. For example:
  - $\pi_{K_n}(x) = x(x-1)(x-2)\dots(x-n+1)$;
  - $\pi_T(x) = x(x-1)^{n-1}$, for any tree T on n vertices.

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Since π_{G-e} and $\pi_{G/e}$ are of degree at most $|V(G)|$, so is π_G. Now, fix a non-negative integer k. We must show that there are precisely $\pi_G(k)$ many k-colorings of G.

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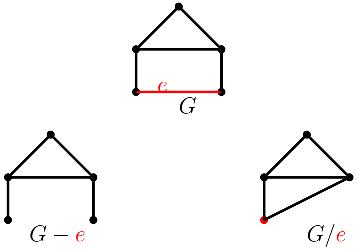

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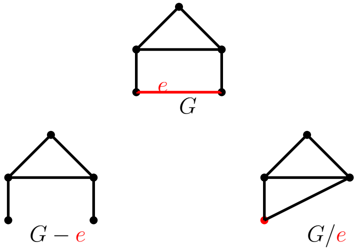

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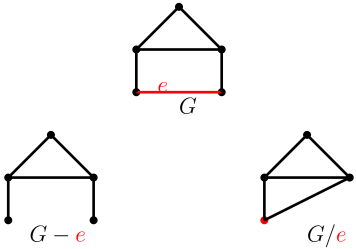

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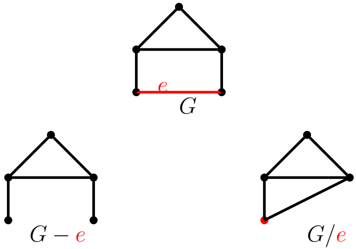

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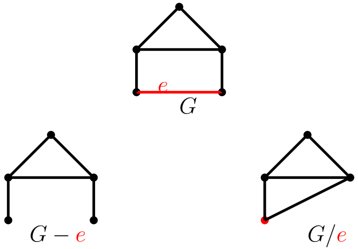

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- Reminder: The polynomial π_G is called the *chromatic polynomial* of G.
- The proof of that lemma in fact gives us a recursive formula for π_G, as follows:
  - if G is edgeless, then $\pi_G(x) = x^{|V(G)|}$;
  - if G has a loop, then $\pi_G(x) = 0$;
  - if G is loopless and has at least one edge, say e, then

$$\pi_G(x) \quad = \quad \pi_{G-e}(x) - \pi_{G/e}(x).$$

Note that $G - e$ and G/e have fewer edges than G, and so our formula really is recursive.

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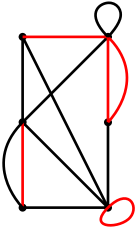

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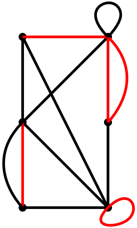

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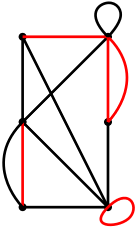

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- In the example above (where the edges of A are in red), we have that $k(G) = 1$, $k_G(A) = 3$, $|A| = 5$, and $|V(G)| = 6$; so, $r_G(A) = 2$ and $c_G(A) = 2$.

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- Since it is NP-hard to compute the chromatic polynomial, it is NP-hard to compute the Tutte polynomial.
- Clearly, if G is edgeless, then $T_G(x, y) = 1$.
- Otherwise, we can get a recursive formula for $T_G(x, y)$, as follows (next slide).

Definition

A *bridge* in a multigraph G is an edge e of G s.t. $G - e$ has more components than G.

Lemma 3.1

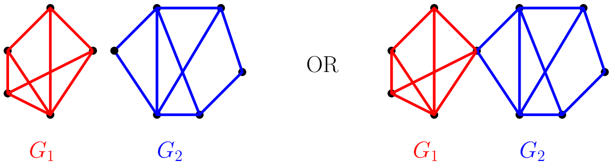
Let e be an edge of a multigraph G. Then

$$T_G(x,y) = \begin{cases} xT_{G/e}(x,y) & \text{if } e \text{ is a bridge of } G \\ yT_{G-e}(x,y) & \text{if } e \text{ is a loop of } G \\ T_{G-e}(x,y) + T_{G/e}(x,y) & \text{otherwise} \end{cases}$$

Proof. Lecture Notes (easy but slightly messy; simply uses the definition).

Lemma 3.2

If multigraphs G_1 and G_2 have at most one vertex and no edges in common, then $T_{G_1 \cup G_2} = T_{G_1} T_{G_2}$.



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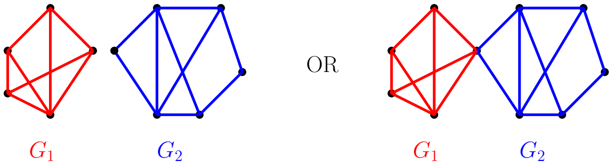

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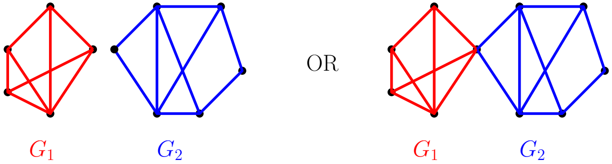

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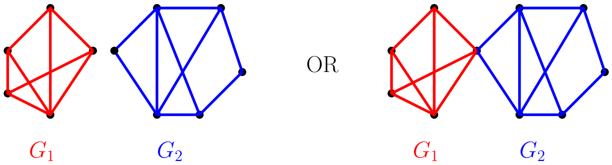

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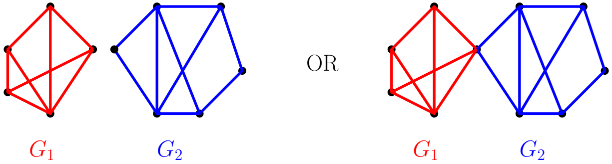

Proof (outline). This follows by induction (on the number of edges) from Lemma 3.1. The details are in the Lecture Notes, but here's an idea. For edgeless (multi)graphs, the Tutte polynomial is 1, and the result is immediate. Now suppose $G := G_1 \cup G_2$ has an edge e; by symmetry, we may assume that $e \in E(G_2)$. Furthermore, $G - e = G_1 \cup (G_2 - e)$ and (if e is not a loop) $G/e = G_1 \cup (G_2/e)$.

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Proof (outline). If e is neither a bridge nor a loop of G, then it is neither a bridge nor a loop of G_2, and so

$$\begin{array}{lll} & T_G(x,y) & \\ = & T_{G-e}(x,y) + T_{G/e}(x,y) & \text{by Lemma 3.1} \\ = & T_{G_1 \cup (G_2 - e)}(x,y) + T_{G_1 \cup (G_2/e)}(x,y) & \\ = & T_{G_1}(x,y) T_{G_2-e}(x,y) + T_{G_1}(x,y) T_{G_2/e}(x,y) & \text{by the ind. hyp.} \\ = & T_{G_1}(x,y) \Big(T_{G_2-e}(x,y) + T_{G/e}(x,y) \Big) & \\ = & T_{G_1}(x,y) T_{G_2}(x,y) & \text{by Lemma 3.1} \end{array}$$

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= \;& T_{G_1}(x,y) T_{G_2}(x,y) & \text{by Lemma 3.1}
\end{aligned}$$

Other cases: Lecture Notes.

Definition

A *block* of a multigraph G is a maximal connected subgraph of G that has no cut-vertices.[a]

[a] A loop with its unique endpoint is considered a block.

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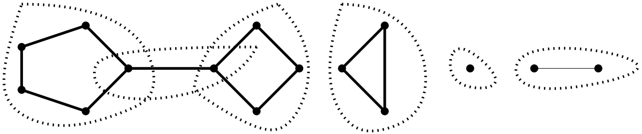
- For example, the (disconnected) graph below has six blocks, in dotted bags.



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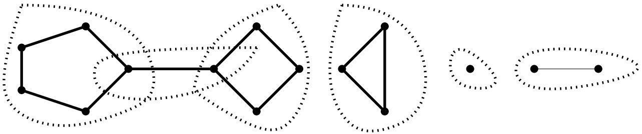

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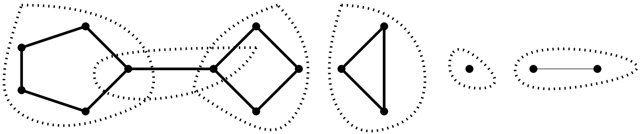

- Note that a (multi)graph can be built from its blocks by repeatedly taking disjoint unions and gluing along single vertices.
- Lemma 3.2 guarantees that the Tutte polynomial of a multigraph G is the product of the Tutte polynomials of its blocks.

Lemma 4.1

Every multigraph G satisfies

$$\pi_G(x) \;=\; (-1)^{|V(G)|-k(G)}\; x^{k(G)}\; T_G(1-x,0).$$

Proof. This follows by induction on the number of edges, using the recursive formulas for the chromatic and Tutte polynomials. Details: Lecture Notes.

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Proof. This follows by induction on the number of edges, using the recursive formulas for the chromatic and Tutte polynomials. Details: Lecture Notes.

- So, if we know the Tutte polynomial of a (multi)graph, then we can easily compute the chromatic polynomial.
- However, computing the Tutte polynomial is hard!

Proposition 5.1

For all multigraphs G, $T_G(2,2) = 2^{|E(G)|}$.

Proposition 5.2

For all multigraphs G, $T_G(2,1)$ is the number of acyclic spanning subgraphs of G.

Proposition 5.3

If G is a connected multigraph, then $T_G(1,2)$ is the number of connected spanning subgraphs of G.

Proposition 5.4

If G is a connected multigraph, then $T_G(1,1)$ is the number of spanning trees of G.

- $r_G(A) := k_G(A) - k(G)$ and $c_G(A) := k_G(A) + |A| - |V(G)|$;
- $T_G(x,y) := \sum_{A \subseteq E(G)} (x-1)^{r_G(A)}(y-1)^{c_G(A)}$.

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For all multigraphs G, $T_G(2,2) = 2^{|E(G)|}$.

Proof. By the definition of the Tutte polynomial, we have that

$$T_G(2,2) = \sum_{A \subseteq E(G)} (2-1)^{r_G(A)}(2-1)^{c_G(A)} = \sum_{A \subseteq E(G)} 1.$$

So, $T_G(2,2)$ is equal to the number of subsets A of $E(G)$, which is precisely $2^{|E(G)|}$.

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For all multigraphs G, $T_G(2,1)$ is the number of acyclic spanning subgraphs of G.

Proof. By the definition of the Tutte polynomial, we have that

$$T_G(2,1) = \sum_{A \subseteq E(G)} (2-1)^{r_G(A)} (1-1)^{c_G(A)} = \sum_{A \subseteq E(G)} 0^{c_G(A)}$$

Now, $0^{c_G(A)} = 1$ if $c_G(A) = 0$, and $0^{c_G(A)} = 0$ otherwise. So, $T_G(2,1)$ is equal to the number of subsets A of $E(G)$ s.t. $c_G(A) = 0$, i.e. $k_G(A) + |A| - |V(G)| = 0$, which is equivalent to $k_G(A) = |V(G)| - |A|$. But this last equality holds precisely when the multigraph $(V(G), A)$ is a forest. The result is now immediate.

- $r_G(A) := k_G(A) - k(G)$ and $c_G(A) := k_G(A) + |A| - |V(G)|$;
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Proposition 5.3

If G is a connected multigraph, then $T_G(1, 2)$ is the number of connected spanning subgraphs of G.

Proof.

- $r_G(A) := k_G(A) - k(G)$ and $c_G(A) := k_G(A) + |A| - |V(G)|$;
- $T_G(x,y) := \sum_{A \subseteq E(G)} (x-1)^{r_G(A)}(y-1)^{c_G(A)}$.

Proposition 5.3

If G is a connected multigraph, then $T_G(1,2)$ is the number of connected spanning subgraphs of G.

Proof. Let G be a connected multigraph. Then by the definition of the Tutte polynomial, we have that

$$T_G(1,2) = \sum_{A \subseteq E(G)} (1-1)^{r_G(A)}(2-1)^{c_G(A)} = \sum_{A \subseteq E(G)} 0^{r_G(A)}$$

Now, $0^{r_G(A)} = 1$ if $r_G(A) = 0$, and $0^{r_G(A)} = 0$ otherwise. So, $T_G(1,2)$ is equal to the number of subsets A of $E(G)$ s.t. $r_G(A) = 0$, i.e. $k_G(A) - k(G) = 0$. Since G is connected, we have that $k(G) = 1$, and so $T_G(1,2)$ is equal to the number of subsets A of $E(G)$ s.t. $k_G(A) = 1$, i.e. to the number of connected spanning subgraphs of G.

- $r_G(A) := k_G(A) - k(G)$ and $c_G(A) := k_G(A) + |A| - |V(G)|$;
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Proposition 5.4

If G is a connected multigraph, then $T_G(1,1)$ is the number of spanning trees of G.

Proof. Let G be a connected multigraph. Then by the definition of the Tutte polynomial, we have that

$$T_G(1,1) = \sum_{A \subseteq E(G)} (1-1)^{r_G(A)}(1-1)^{c_G(A)} = \sum_{A \subseteq E(G)} 0^{r_G(A)+c_G(A)}$$

Now, $0^{r_G(A)+c_G(A)} = 1$ if $r_G(A) + c_G(A) = 0$, and $0^{r_G(A)+c_G(A)} = 0$ otherwise. So, $T_G(1,1)$ is the number of subsets A of $E(G)$ s.t. $r_G(A) = c_G(A) = 0$. But $r_G(A) + c_G(A) = 0$ iff the multigraph $(V(G), A)$ is connected and acyclic (as in the proof of Propositions 5.2 and 5.3). So, $r_G(A) = c_G(A) = 0$ iff $(V(G), A)$ is a tree (equivalently: a spanning tree of G).

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