NDMI012: Combinatorics and Graph Theory 2

Lecture #8

Hamiltonian graphs

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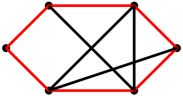
Definition

A *Hamiltonian cycle* (or a *Hamilton cycle*) of a graph G is a cycle of G that passes through all vertices of G.



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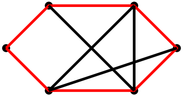

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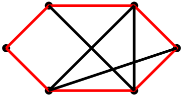

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- It is NP-hard to determine if a graph is Hamiltonian.

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For a real number $t > 0$, a graph G is *t-tough* if for every set $S \subsetneq V(G)$, $G \setminus S$ has at most $\max\{1, \frac{|S|}{t}\}$ components.

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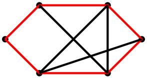

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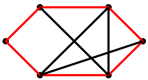

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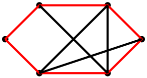

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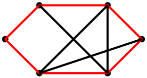

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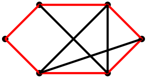

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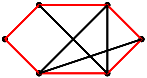

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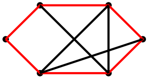

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Lemma 2.1

Let G be a graph, and let x and y be distinct, non-adjacent vertices of G that satisfy $d_G(x) + d_G(y) \geq |V(G)|$. Then G is Hamiltonian iff $G + xy$ is Hamiltonian.

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Suppose now that $G + xy$ is Hamiltonian; we must show that G is Hamiltonian. Let C be a Hamiltonian cycle of $G + xy$. If $xy \notin E(C)$, then C is a Hamiltonian cycle of G, and we are done. So, assume that $xy \in E(C)$.

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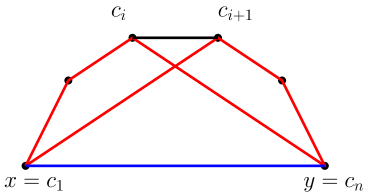

But now $\underbrace{x}_{=c_1}, c_2, \dots, c_i, \underbrace{y}_{=c_n}, c_{n-1}, \dots, c_{i+1}, \underbrace{x}_{=c_1}$ is a Hamiltonian cycle of G, and so G is Hamiltonian.

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The *Chvátal closure* of a graph G is the graph obtained by repeatedly adding edges between non-adjacent vertices x, y s.t. $d(x) + d(y) \geq |V(G)|$, until no more such edges can be added.

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Theorem 2.2

A graph is Hamiltonian iff its Chvátal closure is Hamiltonian.

Proof. This follows from Lemma 2.1 by an easy induction.

Theorem 2.3 [Ore]

Let G be a graph on at least three vertices. Assume that for all distinct, non-adjacent vertices x, y of G, we have that $d_G(x) + d_G(y) \geq |V(G)|$. Then G is Hamiltonian.

Proof. The Chvátal closure of G is the complete graph on $|V(G)|$ vertices, which (since $|V(G)| \geq 3$) is clearly Hamiltonian. So, by Theorem 2.2, G is also Hamiltonian.

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Theorem 2.4 [Dirac]

Let G be a graph on at least three vertices. If $\delta(G) \geq \frac{|V(G)|}{2}$, then G is Hamiltonian.

Proof. This is an immediate corollary of Theorem 2.3.

Definition

Let $\mathbf{a} = (a_1, \ldots, a_n)$ be a list (vector) of integers s.t. $0 \leq a_1 \leq \cdots \leq a_n \leq n-1$. A graph G on n vertices *dominates* $\mathbf{a}$ if for some ordering $v_1, \ldots, v_n$ of the vertices of G, we have that $d_G(v_1) \geq a_1, \ldots, d_G(v_n) \geq a_n$. We say that $\mathbf{a}$ is *Hamiltonian* if every n-vertex graph that dominates $\mathbf{a}$ is Hamiltonian.

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Theorem 2.5

Let $n \geq 3$ be an integer, and let $\mathbf{a} = (a_1, \ldots, a_n)$ be a sequence of integers s.t. $0 \leq a_1 \leq \cdots \leq a_n \leq n-1$. Then the following are equivalent:

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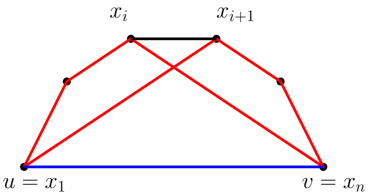
Proof (continued). Let $S := \{i \mid 1 \leq i \leq n-1,\ ux_{i+1} \in E(G)\}$; clearly, $s := |S| = d_G(u)$. If there exists some $i \in S$ s.t. $vx_i \in E(G)$, then $\underbrace{x_1}_{=u}, x_2, \ldots, x_i, \underbrace{x_n}_{=v}, x_{n-1}, \ldots, x_{i+1}, \underbrace{x_1}_{=u}$ would be a Hamiltonian cycle in G, contrary to the fact that G is not Hamiltonian.



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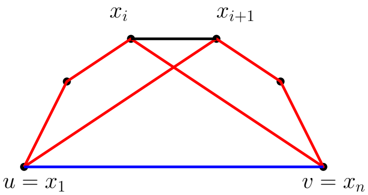

So, no such i exists, and it follows that $d_G(v) \leq n - 1 - s$.

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But now $d_G(u) + d_G(v) \leq s + (n - 1 - s) = n - 1$; since $d_G(u) \leq d_G(v)$, we deduce that $d_G(u) < \frac{n}{2}$, and so $s < \frac{n}{2}$.

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$d_G(v) \leq n-1-s$, v is non-adjacent to all x_i's with $i \in S$.

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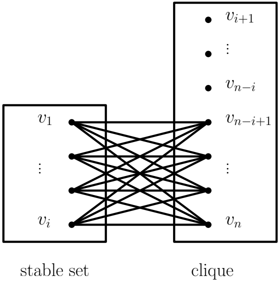

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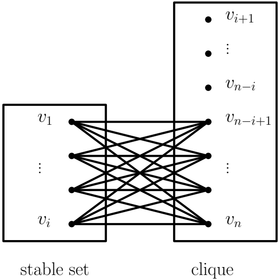

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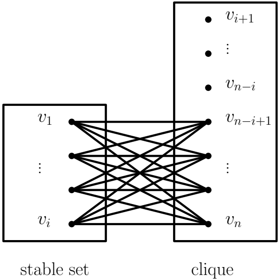

Also, the graph is not 1-tough, because deleting $\{v_{n-i+1}, \ldots, v_n\}$ yields a graph with $i + 1$ components. So, by Proposition 1.2, G is not Hamiltonian, and it follows that (b) does not hold.

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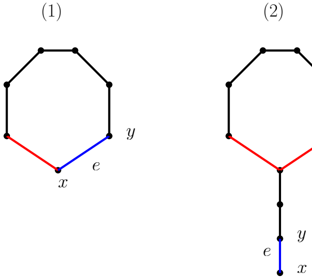
A *lollipop* is a connected subgraph H of G s.t. $V(H) = V(G)$, $e \in E(H)$, and H satisfies one of the following:

(1) H is a cycle;

(2) $d_H(x) = 1$, H has one vertex of degree three, and all other vertices of H are of degree two.

Proof (continued).



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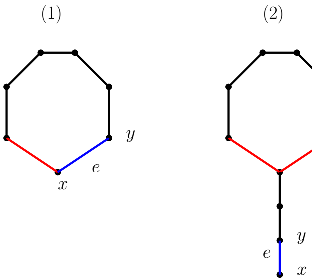

If H is a lollipop that satisfies (1), then H has a unique *tail*, namely the unique edge of H incident with x and distinct from e. On the other hand, if H is a lollipop that satisfies (2), then H has two *tails*, namely, the two edges of the unique cycle of H that are incident with the unique vertex of degree three in H.

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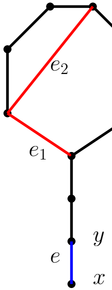
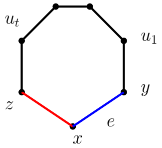


Proof (continued). Suppose that $H = x, y, u_1, \ldots, u_t, z, x$ $(t \geq 0)$ is a lollipop satisfying (1), i.e. H is a Hamiltonian cycle of G containing e.



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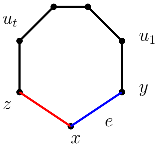

Then xz is the unique tail of H, and the neighbors of H in L are precisely the graphs that can be obtained from $H - xz$ by adding an edge between z and a vertex in $N_G(z) \setminus N_H(z)$.

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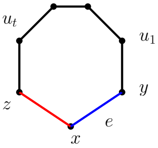
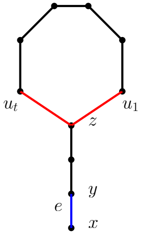


Then xz is the unique tail of H, and the neighbors of H in L are precisely the graphs that can be obtained from $H - xz$ by adding an edge between z and a vertex in $N_G(z) \setminus N_H(z)$. So, $d_L(H) = |N_G(z) \setminus N_H(z)| = d_G(z) - 2$; since $d_G(z)$ is odd, so is $d_L(H)$.

Proof (continued). Suppose now that H is a lollipop satisfying (2); let $z, u_1, \dots, u_t, z$ ($t \geq 2$) be the unique cycle of H, where z is the unique vertex of degree three in H.



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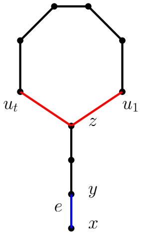

Then the lollipop H has two tails, namely zu_1 and zu_t, and the neighbors of H in L are precisely the graphs that an be obtained in one of the following two ways as follows:

- by starting with $H - zu_1$, and then adding an edge between u_1 and $N_G(u_1) \setminus \{z, u_2\}$;
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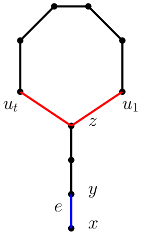

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So, $d_L(H) = (d_G(u_1) - 2) + (d_G(u_t) - 2) = d_G(u_1) + d_G(u_t) - 4$. Since all vertices of G have odd degree, we deduce that $d_L(H)$ is even.

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Proof (continued). We have now shown that the odd-degree vertices of our auxiliary graph L are precisely the Hamiltonian cycles of H that contain the edge e. But clearly, L has an even number of odd-degree vertices (because the sum of degrees in any graph is even), and so the number of Hamiltonian cycles of G containing e is even.