NDMI012: Combinatorics and Graph Theory 2

Lecture #7

Perfect graphs

Irena Penev

April 14, 2021

Definition

A graph is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$.

Definition

A graph is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Definition

A graph is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

- The Perfect Graph Theorem was originally conjectured by Berge (1961).

Definition

A graph is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

- The Perfect Graph Theorem was originally conjectured by Berge (1961).
- Before it was proven, the Perfect Graph Theorem was known as the Weak Perfect Graph Conjecture.

Definition

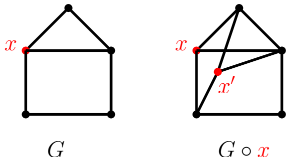
A graph is *perfect* if all its induced subgraphs H satisfy $\chi(H) = \omega(H)$.

The Perfect Graph Theorem [Lovász, 1972]

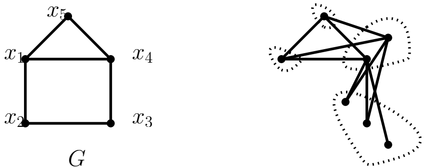
A graph is perfect if and only if its complement is perfect.

- The Perfect Graph Theorem was originally conjectured by Berge (1961).
- Before it was proven, the Perfect Graph Theorem was known as the Weak Perfect Graph Conjecture.
- We will prove the theorem, but first we need some terminology and a lemma.

- *Duplicating a vertex* x of a graph G produces a supergraph $G \circ x$ by adding to G a vertex x' and making it adjacent to all the neighbors of x in G, and to no other vertices of G (in particular, x and x' are nonadjacent in $G \circ x$).



- *Vertex multiplication* of a graph G with vertex set $V(G) = \{x_1, \ldots, x_n\}$ by a nonnegative integer vector $h = (h_1, \ldots, h_n)$ is the graph $G \circ h$ having h_i pairwise nonadjacent copies of x_i, such that copies of x_i and x_j are adjacent in $G \circ x$ if and only if $x_i x_j \in E(G)$.



G

$h = (1, 0, 3, 2, 1)$

$G \circ h$

- A *clique cover* of a graph G is a partition of $V(G)$ into cliques.

- A *clique cover* of a graph G is a partition of $V(G)$ into cliques.
- The *clique cover number* of G, denoted by $\overline{\chi}(G)$, is the smallest size of a clique cover of G.

- A *clique cover* of a graph G is a partition of $V(G)$ into cliques.
- The *clique cover number* of G, denoted by $\overline{\chi}(G)$, is the smallest size of a clique cover of G.
- A *minimum clique cover* of G is a clique cover of size $\overline{\chi}(G)$.

- A *clique cover* of a graph G is a partition of $V(G)$ into cliques.
- The *clique cover number* of G, denoted by $\overline{\chi}(G)$, is the smallest size of a clique cover of G.
- A *minimum clique cover* of G is a clique cover of size $\overline{\chi}(G)$.
- Clearly, $\overline{\chi}(G) = \chi(\overline{G})$ and $\alpha(G) \leq \overline{\chi}(G)$.

- A *clique cover* of a graph G is a partition of $V(G)$ into cliques.
- The *clique cover number* of G, denoted by $\overline{\chi}(G)$, is the smallest size of a clique cover of G.
- A *minimum clique cover* of G is a clique cover of size $\overline{\chi}(G)$.
- Clearly, $\overline{\chi}(G) = \chi(\overline{G})$ and $\alpha(G) \leq \overline{\chi}(G)$.

- Initially, Berge defined two types of perfection, "χ-perfection" and "α-perfection."

- A *clique cover* of a graph G is a partition of $V(G)$ into cliques.
- The *clique cover number* of G, denoted by $\overline{\chi}(G)$, is the smallest size of a clique cover of G.
- A *minimum clique cover* of G is a clique cover of size $\overline{\chi}(G)$.
- Clearly, $\overline{\chi}(G) = \chi(\overline{G})$ and $\alpha(G) \leq \overline{\chi}(G)$.

- Initially, Berge defined two types of perfection, "χ-perfection" and "α-perfection."
  - A graph G is χ*-perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$.
    - So, χ-perfection is the same as perfection.

- A *clique cover* of a graph G is a partition of $V(G)$ into cliques.
- The *clique cover number* of G, denoted by $\overline{\chi}(G)$, is the smallest size of a clique cover of G.
- A *minimum clique cover* of G is a clique cover of size $\overline{\chi}(G)$.
- Clearly, $\overline{\chi}(G) = \chi(\overline{G})$ and $\alpha(G) \leq \overline{\chi}(G)$.

- Initially, Berge defined two types of perfection, "χ-perfection" and "α-perfection."
  - A graph G is *χ-perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$.
    - So, χ-perfection is the same as perfection.
  - A graph G is *α-perfect* if every induced subgraph H of G satisfies $\overline{\chi}(H) = \alpha(H)$.

- A *clique cover* of a graph G is a partition of $V(G)$ into cliques.
- The *clique cover number* of G, denoted by $\overline{\chi}(G)$, is the smallest size of a clique cover of G.
- A *minimum clique cover* of G is a clique cover of size $\overline{\chi}(G)$.
- Clearly, $\overline{\chi}(G) = \chi(\overline{G})$ and $\alpha(G) \leq \overline{\chi}(G)$.

- Initially, Berge defined two types of perfection, "χ-perfection" and "α-perfection."
  - A graph G is χ*-perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$.
    - So, χ-perfection is the same as perfection.
  - A graph G is α*-perfect* if every induced subgraph H of G satisfies $\overline{\chi}(H) = \alpha(H)$.
- Obviously, a graph is χ-perfect (i.e. perfect) if and only if its complement is α-perfect.

Lemma 1.1 [Berge, 1961]

Vertex multiplication preserves χ-perfection and α-perfection.[a]

[a] This means that for every graph G with vertex set $V(G) = \{x_1, \ldots, x_n\}$, and every nonnegative integer vector $h = (h_1, \ldots, h_n)$, we have the following:

- if G is χ-perfect, then so is $G \circ h$;
- if G is α-perfect, then so is $G \circ h$.

Proof.

Claim 1. *Vertex duplication preserves χ-perfection.*

Proof of Claim 1.

Claim 1. *Vertex duplication preserves χ-perfection.*

Proof of Claim 1. Let G be a χ-perfect graph, and assume inductively that any graph obtained by duplicating one vertex of a χ-perfect graph on fewer than $|V(G)|$ vertices is χ-perfect.

Claim 1. *Vertex duplication preserves χ-perfection.*

Proof of Claim 1. Let G be a χ-perfect graph, and assume inductively that any graph obtained by duplicating one vertex of a χ-perfect graph on fewer than $|V(G)|$ vertices is χ-perfect. Let $x \in V(G)$; we must show that $G \circ x$ is χ-perfect. Let x' be the "duplicate" of x in $G \circ x$.

Claim 1. *Vertex duplication preserves χ-perfection.*

Proof of Claim 1. Let G be a χ-perfect graph, and assume inductively that any graph obtained by duplicating one vertex of a χ-perfect graph on fewer than $|V(G)|$ vertices is χ-perfect. Let $x \in V(G)$; we must show that $G \circ x$ is χ-perfect. Let x' be the "duplicate" of x in $G \circ x$. It suffices to show that $\chi(G \circ x) = \omega(G \circ x)$, for the rest follows from the induction hypothesis.

Clearly, we can extend an optimal coloring of G to a proper coloring of $G \circ x$, by giving x' the same color as x. So, $\chi(G \circ x) = \chi(G)$.

Claim 1. *Vertex duplication preserves χ-perfection.*

Proof of Claim 1. Let G be a χ-perfect graph, and assume inductively that any graph obtained by duplicating one vertex of a χ-perfect graph on fewer than $|V(G)|$ vertices is χ-perfect. Let $x \in V(G)$; we must show that $G \circ x$ is χ-perfect. Let x' be the "duplicate" of x in $G \circ x$. It suffices to show that $\chi(G \circ x) = \omega(G \circ x)$, for the rest follows from the induction hypothesis.

Clearly, we can extend an optimal coloring of G to a proper coloring of $G \circ x$, by giving x' the same color as x. So, $\chi(G \circ x) = \chi(G)$. Further, no clique contains both x and x', and it readily follows that $\omega(G \circ x) = \omega(G)$.

Claim 1. *Vertex duplication preserves χ-perfection.*

Proof of Claim 1. Let G be a χ-perfect graph, and assume inductively that any graph obtained by duplicating one vertex of a χ-perfect graph on fewer than $|V(G)|$ vertices is χ-perfect. Let $x \in V(G)$; we must show that $G \circ x$ is χ-perfect. Let x' be the "duplicate" of x in $G \circ x$. It suffices to show that $\chi(G \circ x) = \omega(G \circ x)$, for the rest follows from the induction hypothesis.

Clearly, we can extend an optimal coloring of G to a proper coloring of $G \circ x$, by giving x' the same color as x. So, $\chi(G \circ x) = \chi(G)$. Further, no clique contains both x and x', and it readily follows that $\omega(G \circ x) = \omega(G)$.

Since G is χ-perfect, we have that $\chi(G) = \omega(G)$, and we now see that $\chi(G \circ x) = \chi(G) = \omega(G) = \omega(G \circ x)$. This proves Claim 1.

Claim 2. *Vertex duplication preserves α-perfection.*

Proof of Claim 2.

Claim 2. *Vertex duplication preserves α-perfection.*

Proof of Claim 2. Let G be an α-perfect graph, and assume inductively that any graph obtained by duplicating one vertex of an α-perfect graph on fewer than $|V(G)|$ vertices is α-perfect.

Claim 2. *Vertex duplication preserves α-perfection.*

Proof of Claim 2. Let G be an α-perfect graph, and assume inductively that any graph obtained by duplicating one vertex of an α-perfect graph on fewer than $|V(G)|$ vertices is α-perfect. Let $x \in V(G)$; we must show that $G \circ x$ is α-perfect. Let x' be the "duplicate" of x in $G \circ x$.

Claim 2. *Vertex duplication preserves α-perfection.*

Proof of Claim 2. Let G be an α-perfect graph, and assume inductively that any graph obtained by duplicating one vertex of an α-perfect graph on fewer than $|V(G)|$ vertices is α-perfect. Let $x \in V(G)$; we must show that $G \circ x$ is α-perfect. Let x' be the "duplicate" of x in $G \circ x$. It suffices to show that $\overline{\chi}(G \circ x) = \alpha(G \circ x)$, for the rest follows from the induction hypothesis.

Claim 2. *Vertex duplication preserves α-perfection.*

Proof of Claim 2. Let G be an α-perfect graph, and assume inductively that any graph obtained by duplicating one vertex of an α-perfect graph on fewer than $|V(G)|$ vertices is α-perfect. Let $x \in V(G)$; we must show that $G \circ x$ is α-perfect. Let x' be the "duplicate" of x in $G \circ x$. It suffices to show that $\overline{\chi}(G \circ x) = \alpha(G \circ x)$, for the rest follows from the induction hypothesis.

Suppose first that x belongs to a maximum stable set of G.

Claim 2. *Vertex duplication preserves α-perfection.*

Proof of Claim 2. Let G be an α-perfect graph, and assume inductively that any graph obtained by duplicating one vertex of an α-perfect graph on fewer than $|V(G)|$ vertices is α-perfect. Let $x \in V(G)$; we must show that $G \circ x$ is α-perfect. Let x' be the "duplicate" of x in $G \circ x$. It suffices to show that $\overline{\chi}(G \circ x) = \alpha(G \circ x)$, for the rest follows from the induction hypothesis.

Suppose first that x belongs to a maximum stable set of G. Then $\alpha(G \circ x) = \alpha(G) + 1$.

Claim 2. *Vertex duplication preserves α-perfection.*

Proof of Claim 2. Let G be an α-perfect graph, and assume inductively that any graph obtained by duplicating one vertex of an α-perfect graph on fewer than $|V(G)|$ vertices is α-perfect. Let $x \in V(G)$; we must show that $G \circ x$ is α-perfect. Let x' be the "duplicate" of x in $G \circ x$. It suffices to show that $\overline{\chi}(G \circ x) = \alpha(G \circ x)$, for the rest follows from the induction hypothesis.

Suppose first that x belongs to a maximum stable set of G. Then $\alpha(G \circ x) = \alpha(G) + 1$. Since $\overline{\chi}(G) = \alpha(G)$ (because G is α-perfect), we can obtain a clique cover of size $\alpha(G) + 1$ by adding $\{x'\}$ as a one-vertex clique to some set of $\overline{\chi}(G)$ cliques covering G. This is enough because now we have that $\overline{\chi}(G) + 1 = \alpha(G) + 1 = \alpha(G \circ x) \leq \overline{\chi}(G \circ x) \leq \overline{\chi}(G) + 1$, and so $\overline{\chi}(G \circ x) = \alpha(G \circ x)$.

Claim 2. *Vertex duplication preserves α-perfection.*

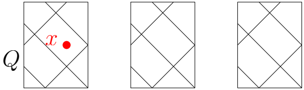
Proof of Claim 2 (continued). We may now assume that x does not belong to any maximum stable set of G.

Claim 2. *Vertex duplication preserves α-perfection.*

Proof of Claim 2 (continued). We may now assume that x does not belong to any maximum stable set of G. Then $\alpha(G \circ x) = \alpha(G)$.

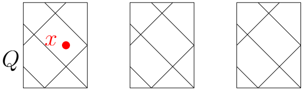
Claim 2. *Vertex duplication preserves α-perfection.*

Proof of Claim 2 (continued). We may now assume that x does not belong to any maximum stable set of G. Then $\alpha(G \circ x) = \alpha(G)$. Let Q be the clique containing x in a minimum clique cover of G.



Claim 2. *Vertex duplication preserves α-perfection.*

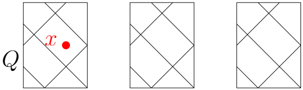
Proof of Claim 2 (continued). We may now assume that x does not belong to any maximum stable set of G. Then $\alpha(G \circ x) = \alpha(G)$. Let Q be the clique containing x in a minimum clique cover of G.



Since $\overline{\chi}(G) = \alpha(G)$, Q intersects every maximum stable set of G.

Claim 2. *Vertex duplication preserves α-perfection.*

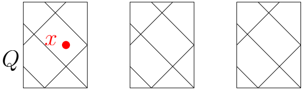
Proof of Claim 2 (continued). We may now assume that x does not belong to any maximum stable set of G. Then $\alpha(G \circ x) = \alpha(G)$. Let Q be the clique containing x in a minimum clique cover of G.



Since $\overline{\chi}(G) = \alpha(G)$, Q intersects every maximum stable set of G. Since x belongs to no maximum stable set, $Q' = Q \setminus \{x\}$ also intersects every maximum stable set, and hence
$\alpha(G \setminus Q') = \alpha(G) - 1$.

Claim 2. *Vertex duplication preserves α-perfection.*

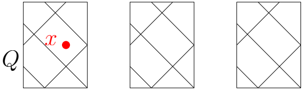
Proof of Claim 2 (continued). We may now assume that x does not belong to any maximum stable set of G. Then $\alpha(G \circ x) = \alpha(G)$. Let Q be the clique containing x in a minimum clique cover of G.



Since $\overline{\chi}(G) = \alpha(G)$, Q intersects every maximum stable set of G. Since x belongs to no maximum stable set, $Q' = Q \setminus \{x\}$ also intersects every maximum stable set, and hence $\alpha(G \setminus Q') = \alpha(G) - 1$. Since G is α-perfect, $\overline{\chi}(G \setminus Q') = \alpha(G \setminus Q')$. To a set of $\alpha(G) - 1$ cliques covering $G \setminus Q'$, add the clique $Q' \cup \{x'\}$ to obtain a set of $\alpha(G) = \alpha(G \circ x)$ cliques covering $G \circ x$;

Claim 2. *Vertex duplication preserves α-perfection.*

Proof of Claim 2 (continued). We may now assume that x does not belong to any maximum stable set of G. Then $\alpha(G \circ x) = \alpha(G)$. Let Q be the clique containing x in a minimum clique cover of G.



Since $\overline{\chi}(G) = \alpha(G)$, Q intersects every maximum stable set of G. Since x belongs to no maximum stable set, $Q' = Q \setminus \{x\}$ also intersects every maximum stable set, and hence $\alpha(G \setminus Q') = \alpha(G) - 1$. Since G is α-perfect, $\overline{\chi}(G \setminus Q') = \alpha(G \setminus Q')$. To a set of $\alpha(G) - 1$ cliques covering $G \setminus Q'$, add the clique $Q' \cup \{x'\}$ to obtain a set of $\alpha(G) = \alpha(G \circ x)$ cliques covering $G \circ x$; we now have that $\overline{\chi}(G \circ x) = \alpha(G \circ x)$. This proves Claim 2.

Lemma 1.1 [Berge, 1961]

Vertex multiplication preserves χ-perfection and α-perfection.

Proof (continued).

Lemma 1.1 [Berge, 1961]

Vertex multiplication preserves χ-perfection and α-perfection.

Proof (continued). So far, we have proven:

> ***Claim 1.*** *Vertex duplication preserves χ-perfection.*
>
> ***Claim 2.*** *Vertex duplication preserves α-perfection.*

Lemma 1.1 [Berge, 1961]

Vertex multiplication preserves χ-perfection and α-perfection.

Proof (continued). Let G be a graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$, and let $h = (h_1, \ldots, h_n)$ be a nonnegative integer vector.

Lemma 1.1 [Berge, 1961]

Vertex multiplication preserves χ-perfection and α-perfection.

Proof (continued). Let G be a graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$, and let $h = (h_1, \ldots, h_n)$ be a nonnegative integer vector. Let A be the set of vertices x_i for which $h_i > 0$.

Lemma 1.1 [Berge, 1961]

Vertex multiplication preserves χ-perfection and α-perfection.

Proof (continued). Let G be a graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$, and let $h = (h_1, \ldots, h_n)$ be a nonnegative integer vector. Let A be the set of vertices x_i for which $h_i > 0$. Clearly, if G is χ-perfect (resp. α-perfect), then $G[A]$ is also χ-perfect (resp. α-perfect).

Lemma 1.1 [Berge, 1961]

Vertex multiplication preserves χ-perfection and α-perfection.

Proof (continued). Let G be a graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$, and let $h = (h_1, \ldots, h_n)$ be a nonnegative integer vector. Let A be the set of vertices x_i for which $h_i > 0$. Clearly, if G is χ-perfect (resp. α-perfect), then $G[A]$ is also χ-perfect (resp. α-perfect).

Now, $G \circ h$ can be obtained from $G[A]$ by a sequence of vertex duplications: if every h_i is 0 or 1 then $G \circ h = G[A]$, and otherwise, $G \circ h$ can be obtained from $G[A]$ by repeatedly duplicating vertices until there are h_i copies of each x_i.

Lemma 1.1 [Berge, 1961]

Vertex multiplication preserves χ-perfection and α-perfection.

Proof (continued). Let G be a graph with vertex set $V(G) = \{x_1, \dots, x_n\}$, and let $h = (h_1, \dots, h_n)$ be a nonnegative integer vector. Let A be the set of vertices x_i for which $h_i > 0$. Clearly, if G is χ-perfect (resp. α-perfect), then $G[A]$ is also χ-perfect (resp. α-perfect).

Now, $G \circ h$ can be obtained from $G[A]$ by a sequence of vertex duplications: if every h_i is 0 or 1 then $G \circ h = G[A]$, and otherwise, $G \circ h$ can be obtained from $G[A]$ by repeatedly duplicating vertices until there are h_i copies of each x_i.

Since vertex duplication preserves χ-perfection and α-perfection (by Claims 1 and 2), an easy induction now guarantees that if G is χ-perfect (resp. α-perfect), then $G \circ h$ is also χ-perfect (resp. α-perfect). This completes the argument.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof. Obviously, it is enough to prove that if a graph is perfect, then so is its complement.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof. Obviously, it is enough to prove that if a graph is perfect, then so is its complement. For this, it is in fact enough to prove that every α-perfect graph is χ-perfect, for then we will have the following sequence of implications for each graph G:

$$G \text{ is } (\chi\text{-})\text{perfect} \implies \overline{G} \text{ is } \alpha\text{-perfect} \implies \overline{G} \text{ is } (\chi\text{-})\text{perfect},$$

which is what we need.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof. Obviously, it is enough to prove that if a graph is perfect, then so is its complement. For this, it is in fact enough to prove that every α-perfect graph is χ-perfect, for then we will have the following sequence of implications for each graph G:

$$G \text{ is } (\chi\text{-})\text{perfect} \implies \overline{G} \text{ is } \alpha\text{-perfect} \implies \overline{G} \text{ is } (\chi\text{-})\text{perfect},$$

which is what we need.

Now, fix an α-perfect graph G, and assume inductively that all α-perfect graphs on fewer than $|V(G)|$ vertices are χ-perfect. We must show that G is χ-perfect.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof. Obviously, it is enough to prove that if a graph is perfect, then so is its complement. For this, it is in fact enough to prove that every α-perfect graph is χ-perfect, for then we will have the following sequence of implications for each graph G:

$$G \text{ is } (\chi\text{-})\text{perfect} \implies \overline{G} \text{ is } \alpha\text{-perfect} \implies \overline{G} \text{ is } (\chi\text{-})\text{perfect},$$

which is what we need.

Now, fix an α-perfect graph G, and assume inductively that all α-perfect graphs on fewer than $|V(G)|$ vertices are χ-perfect. We must show that G is χ-perfect. In view of the induction hypothesis, it suffices to show that $\chi(G) = \omega(G)$.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued).

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued). Reminder: G is α-perfect.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued). Reminder: G is α-perfect.

Suppose first that G has a stable set S that intersects every maximum clique of G.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued). Reminder: G is α-perfect.

Suppose first that G has a stable set S that intersects every maximum clique of G. Then by the minimality of G, $\chi(G \setminus S) = \omega(G \setminus S) = \omega(G) - 1$.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued). Reminder: G is α-perfect.

Suppose first that G has a stable set S that intersects every maximum clique of G. Then by the minimality of G, $\chi(G \setminus S) = \omega(G \setminus S) = \omega(G) - 1$. But now $\chi(G) = \omega(G)$, since we can properly color $G \setminus S$ with $\omega(G) - 1$ colors, and then color all vertices of S with the same new color.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued). Reminder: G is α-perfect.

Suppose first that G has a stable set S that intersects every maximum clique of G. Then by the minimality of G, $\chi(G \setminus S) = \omega(G \setminus S) = \omega(G) - 1$. But now $\chi(G) = \omega(G)$, since we can properly color $G \setminus S$ with $\omega(G) - 1$ colors, and then color all vertices of S with the same new color.

From now on, we assume that every stable set S of G misses (i.e. has an empty intersection with) some maximum clique $Q(S)$; our goal is to derive a contradiction.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued). Reminder: G is α-perfect.

Suppose first that G has a stable set S that intersects every maximum clique of G. Then by the minimality of G, $\chi(G \setminus S) = \omega(G \setminus S) = \omega(G) - 1$. But now $\chi(G) = \omega(G)$, since we can properly color $G \setminus S$ with $\omega(G) - 1$ colors, and then color all vertices of S with the same new color.

From now on, we assume that every stable set S of G misses (i.e. has an empty intersection with) some maximum clique $Q(S)$; our goal is to derive a contradiction. Set $V(G) = \{x_1, \dots, x_n\}$, and let $\mathcal{S} = \{S_1, \dots, S_t\}$ be the set of all maximal stable sets of G.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued). Reminder: G is α-perfect.

Suppose first that G has a stable set S that intersects every maximum clique of G. Then by the minimality of G, $\chi(G \setminus S) = \omega(G \setminus S) = \omega(G) - 1$. But now $\chi(G) = \omega(G)$, since we can properly color $G \setminus S$ with $\omega(G) - 1$ colors, and then color all vertices of S with the same new color.

From now on, we assume that every stable set S of G misses (i.e. has an empty intersection with) some maximum clique $Q(S)$; our goal is to derive a contradiction. Set $V(G) = \{x_1, \dots, x_n\}$, and let $\mathcal{S} = \{S_1, \dots, S_t\}$ be the set of all maximal stable sets of G. For every vertex x_j, let h_j be the number of stable sets S in $\mathcal{S}$ such that $x_j \in Q(S)$.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued). Reminder: G is α-perfect.

Suppose first that G has a stable set S that intersects every maximum clique of G. Then by the minimality of G, $\chi(G \setminus S) = \omega(G \setminus S) = \omega(G) - 1$. But now $\chi(G) = \omega(G)$, since we can properly color $G \setminus S$ with $\omega(G) - 1$ colors, and then color all vertices of S with the same new color.

From now on, we assume that every stable set S of G misses (i.e. has an empty intersection with) some maximum clique $Q(S)$; our goal is to derive a contradiction. Set $V(G) = \{x_1, \dots, x_n\}$, and let $\mathcal{S} = \{S_1, \dots, S_t\}$ be the set of all maximal stable sets of G. For every vertex x_j, let h_j be the number of stable sets S in $\mathcal{S}$ such that $x_j \in Q(S)$. Set $h := (h_1, \dots, h_n)$. By Lemma 1.1, $H := G \circ h$ is α-perfect, and so $\overline{\chi}(H) = \alpha(H)$.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

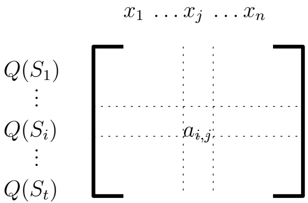
Proof (continued). Reminder: G is α-perfect; h_j is the number of S_i's such that $x_j \in Q(S)$; $H = G \circ h$.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued). Reminder: G is α-perfect; h_j is the number of S_i's such that $x_j \in Q(S)$; $H = G \circ h$.

Let $A = [a_{i,j}]_{t \times n}$ be a 0,1-matrix of the incidence relation between the set of $Q(S)$'s for $S \in \mathcal{S}$ and $V(G)$. So, $a_{i,j} = 1$ if and only if $x_j \in Q(S_i)$.

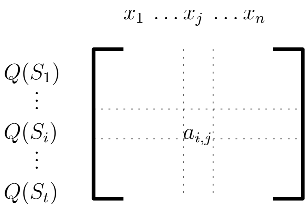


By construction, h_j is the number of 1's in column j of A, and $|V(H)|$ is the total number of 1's in A.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

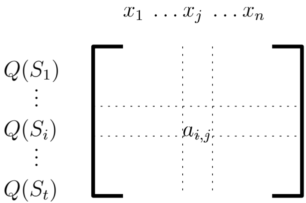
Proof (continued). Reminder: G is α-perfect; h_j is the number of S_i's such that $x_j \in Q(S)$; $H = G \circ h$.



The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued). Reminder: G is α-perfect; h_j is the number of S_i's such that $x_j \in Q(S)$; $H = G \circ h$.

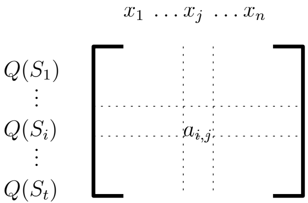


Since each row contributes $\omega(G)$ ones, we have $|V(H)| = \omega(G)|\mathcal{S}|$.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued). Reminder: G is α-perfect; h_j is the number of S_i's such that $x_j \in Q(S)$; $H = G \circ h$.

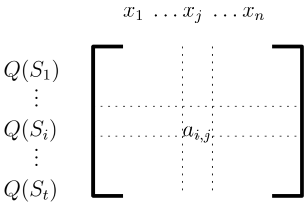


Since each row contributes $\omega(G)$ ones, we have $|V(H)| = \omega(G)|\mathcal{S}|$. Since vertex duplication cannot enlarge cliques, we have $\omega(H) \leq \omega(G)$.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued). Reminder: G is α-perfect; h_j is the number of S_i's such that $x_j \in Q(S)$; $H = G \circ h$.

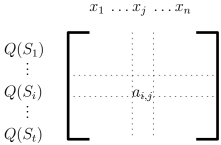


Since each row contributes $\omega(G)$ ones, we have $|V(H)| = \omega(G)|\mathcal{S}|$. Since vertex duplication cannot enlarge cliques, we have $\omega(H) \leq \omega(G)$. Therefore $\overline{\chi}(H) \geq \frac{|V(H)|}{\omega(H)} \geq \frac{|V(H)|}{\omega(G)} = |\mathcal{S}|$.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

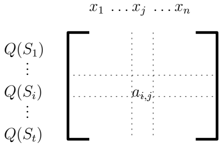
Proof (continued). Reminder: G is α-perfect; h_j is the number of S_i's such that $x_j \in Q(S)$; $H = G \circ h$.



The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued). Reminder: G is α-perfect; h_j is the number of S_i's such that $x_j \in Q(S)$; $H = G \circ h$.

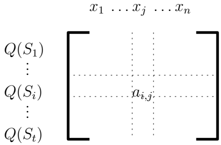


Every stable set of H consists of copies of elements in some stable set of G; so, a maximum stable set of H consists of all copies of all vertices in some maximal stable set of G.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued). Reminder: G is α-perfect; h_j is the number of S_i's such that $x_j \in Q(S)$; $H = G \circ h$.

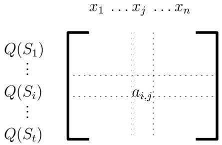


Every stable set of H consists of copies of elements in some stable set of G; so, a maximum stable set of H consists of all copies of all vertices in some maximal stable set of G. Consequently,
$\alpha(H) = \max_{T \in \mathcal{S}} \sum_{j : x_j \in T} h_j$.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued). Reminder: G is α-perfect; h_j is the number of S_i's such that $x_j \in Q(S)$; $H = G \circ h$.



Every stable set of H consists of copies of elements in some stable set of G; so, a maximum stable set of H consists of all copies of all vertices in some maximal stable set of G. Consequently,

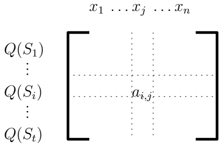
$$\alpha(H) = \max_{T \in \mathcal{S}} \sum_{j : x_j \in T} h_j.$$

The sum above counts the 1's in A that appear in the columns indexed by T.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued). Reminder: G is α-perfect; h_j is the number of S_i's such that $x_j \in Q(S)$; $H = G \circ h$.



Every stable set of H consists of copies of elements in some stable set of G; so, a maximum stable set of H consists of all copies of all vertices in some maximal stable set of G. Consequently,

$$\alpha(H) = \max_{T \in \mathcal{S}} \sum_{j : x_j \in T} h_j.$$

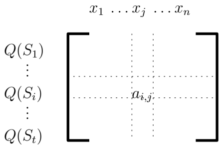
The sum above counts the 1's in A that appear in the columns indexed by T. If we count these 1's by rows, we get

$$\alpha(H) = \max_{T \in \mathcal{S}} \sum_{S \in \mathcal{S}} |T \cap Q(S)|.$$

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

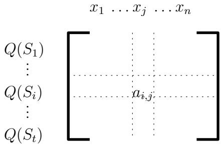
Proof (continued). Reminder: G is α-perfect; h_j is the number of S_i's such that $x_j \in Q(S)$; $H = G \circ h$; $\alpha(H) = \max_{T \in \mathcal{S}} \sum_{S \in \mathcal{S}} |T \cap Q(S)|$.



The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued). Reminder: G is α-perfect; h_j is the number of S_i's such that $x_j \in Q(S)$; $H = G \circ h$; $\alpha(H) = \max_{T \in \mathcal{S}} \sum_{S \in \mathcal{S}} |T \cap Q(S)|$.

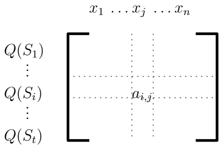


Since T is a stable set, it has at most one vertex in each chosen clique $Q(S)$.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued). Reminder: G is α-perfect; h_j is the number of S_i's such that $x_j \in Q(S)$; $H = G \circ h$; $\alpha(H) = \max_{T \in \mathcal{S}} \sum_{S \in \mathcal{S}} |T \cap Q(S)|$.

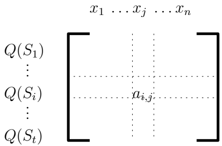


Since T is a stable set, it has at most one vertex in each chosen clique $Q(S)$. Also, $T \cap Q(T) = \emptyset$.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued). Reminder: G is α-perfect; h_j is the number of S_i's such that $x_j \in Q(S)$; $H = G \circ h$; $\alpha(H) = \max_{T \in \mathcal{S}} \sum_{S \in \mathcal{S}} |T \cap Q(S)|$.

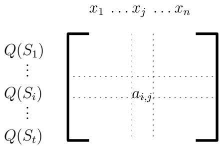


Since T is a stable set, it has at most one vertex in each chosen clique $Q(S)$. Also, $T \cap Q(T) = \emptyset$. So, $|T \cap Q(S)| \leq 1$ for every $S \in \mathcal{S}$, and $|T \cap Q(T)| = 0$.

The Perfect Graph Theorem [Lovász, 1972]

A graph is perfect if and only if its complement is perfect.

Proof (continued). Reminder: G is α-perfect; h_j is the number of S_i's such that $x_j \in Q(S)$; $H = G \circ h$; $\alpha(H) = \max_{T \in \mathcal{S}} \sum_{S \in \mathcal{S}} |T \cap Q(S)|$.



Since T is a stable set, it has at most one vertex in each chosen clique $Q(S)$. Also, $T \cap Q(T) = \emptyset$. So, $|T \cap Q(S)| \leq 1$ for every $S \in \mathcal{S}$, and $|T \cap Q(T)| = 0$. It follows that $\alpha(H) \leq |\mathcal{S}| - 1$. Therefore $\alpha(H) < \overline{\chi}(H)$, contrary to the fact that H is α-perfect.

- A *partial order* of a set X is a binary relation on X that is reflexive, antisymmetric, and transitive.

- A *partial order* of a set X is a binary relation on X that is reflexive, antisymmetric, and transitive.
- A *partially ordered set* (or *poset*) is an ordered pair $(X, \preceq)$ such that X is a set and $\preceq$ is a partial order on X.

- A *partial order* of a set X is a binary relation on X that is reflexive, antisymmetric, and transitive.
- A *partially ordered set* (or *poset*) is an ordered pair $(X, \preceq)$ such that X is a set and $\preceq$ is a partial order on X.
- A *maximal* element of $(X, \preceq)$ is $x \in X$ such that no $y \in X \setminus \{x\}$ satisfies $x \preceq y$.

- A *partial order* of a set X is a binary relation on X that is reflexive, antisymmetric, and transitive.
- A *partially ordered set* (or *poset*) is an ordered pair $(X, \preceq)$ such that X is a set and $\preceq$ is a partial order on X.
- A *maximal* element of $(X, \preceq)$ is $x \in X$ such that no $y \in X \setminus \{x\}$ satisfies $x \preceq y$.
- A *minimal* element of $(X, \preceq)$ is $x \in X$ such that no $y \in X \setminus \{x\}$ satisfies $y \preceq x$.

- A *partial order* of a set X is a binary relation on X that is reflexive, antisymmetric, and transitive.
- A *partially ordered set* (or *poset*) is an ordered pair $(X, \preceq)$ such that X is a set and $\preceq$ is a partial order on X.
- A *maximal* element of $(X, \preceq)$ is $x \in X$ such that no $y \in X \setminus \{x\}$ satisfies $x \preceq y$.
- A *minimal* element of $(X, \preceq)$ is $x \in X$ such that no $y \in X \setminus \{x\}$ satisfies $y \preceq x$.
- We say that $x, y \in X$ are *comparable* with respect to $\preceq$ if either $x \preceq y$ or $y \preceq x$; two elements of X are *incomparable* with respect to $\preceq$ if they are not comparable with respect to $\preceq$.

- A *partial order* of a set X is a binary relation on X that is reflexive, antisymmetric, and transitive.
- A *partially ordered set* (or *poset*) is an ordered pair $(X, \preceq)$ such that X is a set and $\preceq$ is a partial order on X.
- A *maximal* element of $(X, \preceq)$ is $x \in X$ such that no $y \in X \setminus \{x\}$ satisfies $x \preceq y$.
- A *minimal* element of $(X, \preceq)$ is $x \in X$ such that no $y \in X \setminus \{x\}$ satisfies $y \preceq x$.
- We say that $x, y \in X$ are *comparable* with respect to $\preceq$ if either $x \preceq y$ or $y \preceq x$; two elements of X are *incomparable* with respect to $\preceq$ if they are not comparable with respect to $\preceq$.
- A *chain* in $(X, \preceq)$ is a set $C \subseteq X$ such that any two elements of C are comparable with respect to $\preceq$.

- A *partial order* of a set X is a binary relation on X that is reflexive, antisymmetric, and transitive.
- A *partially ordered set* (or *poset*) is an ordered pair $(X, \preceq)$ such that X is a set and $\preceq$ is a partial order on X.
- A *maximal* element of $(X, \preceq)$ is $x \in X$ such that no $y \in X \setminus \{x\}$ satisfies $x \preceq y$.
- A *minimal* element of $(X, \preceq)$ is $x \in X$ such that no $y \in X \setminus \{x\}$ satisfies $y \preceq x$.
- We say that $x, y \in X$ are *comparable* with respect to $\preceq$ if either $x \preceq y$ or $y \preceq x$; two elements of X are *incomparable* with respect to $\preceq$ if they are not comparable with respect to $\preceq$.
- A *chain* in $(X, \preceq)$ is a set $C \subseteq X$ such that any two elements of C are comparable with respect to $\preceq$.
- A *chain decomposition* of $(X, \preceq)$ is a partition of X into chains of $(X, \preceq)$.

- A *partial order* of a set X is a binary relation on X that is reflexive, antisymmetric, and transitive.
- A *partially ordered set* (or *poset*) is an ordered pair $(X, \preceq)$ such that X is a set and $\preceq$ is a partial order on X.
- A *maximal* element of $(X, \preceq)$ is $x \in X$ such that no $y \in X \setminus \{x\}$ satisfies $x \preceq y$.
- A *minimal* element of $(X, \preceq)$ is $x \in X$ such that no $y \in X \setminus \{x\}$ satisfies $y \preceq x$.
- We say that $x, y \in X$ are *comparable* with respect to $\preceq$ if either $x \preceq y$ or $y \preceq x$; two elements of X are *incomparable* with respect to $\preceq$ if they are not comparable with respect to $\preceq$.
- A *chain* in $(X, \preceq)$ is a set $C \subseteq X$ such that any two elements of C are comparable with respect to $\preceq$.
- A *chain decomposition* of $(X, \preceq)$ is a partition of X into chains of $(X, \preceq)$.
- An *antichain* in $(X, \preceq)$ is a set $A \subseteq X$ such that no two elements of A are comparable with respect to $\preceq$.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof. Let $(X, \preceq)$ be a finite partially ordered set, and assume inductively that the theorem is true for smaller partially ordered sets.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof. Let $(X, \preceq)$ be a finite partially ordered set, and assume inductively that the theorem is true for smaller partially ordered sets. We may assume that $X \neq \emptyset$, for otherwise, the result is immediate.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof. Let $(X, \preceq)$ be a finite partially ordered set, and assume inductively that the theorem is true for smaller partially ordered sets. We may assume that $X \neq \emptyset$, for otherwise, the result is immediate. First, it is clear that if $(X, \preceq)$ has an antichain of size k, then no chain decomposition of $(X, \preceq)$ is of size smaller than k (this is because no chain can contain two elements of an antichain).

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof. Let $(X, \preceq)$ be a finite partially ordered set, and assume inductively that the theorem is true for smaller partially ordered sets. We may assume that $X \neq \emptyset$, for otherwise, the result is immediate. First, it is clear that if $(X, \preceq)$ has an antichain of size k, then no chain decomposition of $(X, \preceq)$ is of size smaller than k (this is because no chain can contain two elements of an antichain). It remains to exhibit an antichain of $(X, \preceq)$ and a chain decomposition of $(X, \preceq)$ of the same size.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued).

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Since $(X, \preceq)$ is a nonempty, finite partial order, we see that $(X, \preceq)$ has a maximal element, say x_0.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Since $(X, \preceq)$ is a nonempty, finite partial order, we see that $(X, \preceq)$ has a maximal element, say x_0. Set $X_0 := X \setminus \{x_0\}$, and let A_0 be a maximum antichain in $(X_0, \preceq)$; set $k := |A_0|$.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Since $(X, \preceq)$ is a nonempty, finite partial order, we see that $(X, \preceq)$ has a maximal element, say x_0. Set $X_0 := X \setminus \{x_0\}$, and let A_0 be a maximum antichain in $(X_0, \preceq)$; set $k := |A_0|$. By the induction hypothesis, $(X_0, \preceq)$ has a chain decomposition of size k, say $\{C_1, \dots, C_k\}$.

> ***Claim 1.*** *Any antichain of size k in $(X_0, \preceq)$ intersects each of $C_1, \dots, C_k$ in exactly one element.*

Proof of Claim 1.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Since $(X, \preceq)$ is a nonempty, finite partial order, we see that $(X, \preceq)$ has a maximal element, say x_0. Set $X_0 := X \setminus \{x_0\}$, and let A_0 be a maximum antichain in $(X_0, \preceq)$; set $k := |A_0|$. By the induction hypothesis, $(X_0, \preceq)$ has a chain decomposition of size k, say $\{C_1, \dots, C_k\}$.

> ***Claim 1.*** *Any antichain of size k in $(X_0, \preceq)$ intersects each of $C_1, \dots, C_k$ in exactly one element.*

Proof of Claim 1. Let B be a antichain of size k in $(X_0, \preceq)$.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Since $(X, \preceq)$ is a nonempty, finite partial order, we see that $(X, \preceq)$ has a maximal element, say x_0. Set $X_0 := X \setminus \{x_0\}$, and let A_0 be a maximum antichain in $(X_0, \preceq)$; set $k := |A_0|$. By the induction hypothesis, $(X_0, \preceq)$ has a chain decomposition of size k, say $\{C_1, \dots, C_k\}$.

> ***Claim 1.*** *Any antichain of size k in $(X_0, \preceq)$ intersects each of $C_1, \dots, C_k$ in exactly one element.*

Proof of Claim 1. Let B be a antichain of size k in $(X_0, \preceq)$. Since B is a antichain and $C_1, \dots, C_k$ are chains in $(X_0, \preceq)$, we see that B intersects each of $C_1, \dots, C_k$ in at most one element.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Since $(X, \preceq)$ is a nonempty, finite partial order, we see that $(X, \preceq)$ has a maximal element, say x_0. Set $X_0 := X \setminus \{x_0\}$, and let A_0 be a maximum antichain in $(X_0, \preceq)$; set $k := |A_0|$. By the induction hypothesis, $(X_0, \preceq)$ has a chain decomposition of size k, say $\{C_1, \dots, C_k\}$.

> ***Claim 1.*** *Any antichain of size k in $(X_0, \preceq)$ intersects each of $C_1, \dots, C_k$ in exactly one element.*

Proof of Claim 1. Let B be a antichain of size k in $(X_0, \preceq)$. Since B is a antichain and $C_1, \dots, C_k$ are chains in $(X_0, \preceq)$, we see that B intersects each of $C_1, \dots, C_k$ in at most one element. But since $B \subseteq C_1 \cup \dots \cup C_k$, and since $|B| = k$, we see that B intersects each of $C_1, \dots, C_k$ in exactly one element. This proves Claim 1.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued).

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Now, for all $i \in \{1, \dots, k\}$, let C_i' be the set of all elements of C_i that belong to some antichain of $(X_0, \preceq)$ of size k;

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Now, for all $i \in \{1, \dots, k\}$, let C_i' be the set of all elements of C_i that belong to some antichain of $(X_0, \preceq)$ of size k; then $C_i' \neq \emptyset$ (because, by Claim 1, $C_i \cap A_0 \neq \emptyset$), and we deduce that C_i' has a unique maximal element, call it x_i.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Now, for all $i \in \{1, \dots, k\}$, let C_i' be the set of all elements of C_i that belong to some antichain of $(X_0, \preceq)$ of size k; then $C_i' \neq \emptyset$ (because, by Claim 1, $C_i \cap A_0 \neq \emptyset$), and we deduce that C_i' has a unique maximal element, call it x_i.

Claim 2. *$\{x_1, \dots, x_k\}$ is an antichain in $(X_0, \preceq)$.*

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued).

> ***Claim 2.*** *$\{x_1, \ldots, x_k\}$ is an antichain in $(X_0, \preceq)$.*

Proof of Claim 2.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued).

> ***Claim 2.*** $\{x_1, \dots, x_k\}$ *is an antichain in* $(X_0, \preceq)$.

Proof of Claim 2. We may assume that $k \geq 2$, for otherwise, this is immediate.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued).

Claim 2. *$\{x_1, \dots, x_k\}$ is an antichain in $(X_0, \preceq)$.*

Proof of Claim 2. We may assume that $k \geq 2$, for otherwise, this is immediate. By symmetry, it suffices to show that x_1 and x_2 are incomparable.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued).

Claim 2. $\{x_1, \dots, x_k\}$ *is an antichain in* $(X_0, \preceq)$.

Proof of Claim 2. We may assume that $k \geq 2$, for otherwise, this is immediate. By symmetry, it suffices to show that x_1 and x_2 are incomparable. Let A_1 be an antichain of size k in $(X_0, \preceq)$ such that $x_1 \in A_1 \cap C_1$. By Claim 1, $|A_1 \cap C_2| = 1$; set $A_1 \cap C_2 = \{x_2'\}$. Then $x_2' \in C_2'$, and so (since x_2 is a maximal element of the chain C_2') we have that $x_2' \preceq x_2$.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued).

> ***Claim 2.*** $\{x_1, \dots, x_k\}$ *is an antichain in* $(X_0, \preceq)$.

Proof of Claim 2. We may assume that $k \geq 2$, for otherwise, this is immediate. By symmetry, it suffices to show that x_1 and x_2 are incomparable. Let A_1 be an antichain of size k in $(X_0, \preceq)$ such that $x_1 \in A_1 \cap C_1$. By Claim 1, $|A_1 \cap C_2| = 1$; set $A_1 \cap C_2 = \{x_2'\}$. Then $x_2' \in C_2'$, and so (since x_2 is a maximal element of the chain C_2') we have that $x_2' \preceq x_2$. Now, if $x_2 \preceq x_1$, then by the transitivity of $\preceq$, we have that $x_2' \preceq x_1$, which is impossible since x_1 and x_2' are distinct elements of the antichain A_1. So, $x_2 \not\preceq x_1$.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued).

> ***Claim 2.*** $\{x_1, \dots, x_k\}$ *is an antichain in* $(X_0, \preceq)$.

Proof of Claim 2. We may assume that $k \geq 2$, for otherwise, this is immediate. By symmetry, it suffices to show that x_1 and x_2 are incomparable. Let A_1 be an antichain of size k in $(X_0, \preceq)$ such that $x_1 \in A_1 \cap C_1$. By Claim 1, $|A_1 \cap C_2| = 1$; set $A_1 \cap C_2 = \{x_2'\}$. Then $x_2' \in C_2'$, and so (since x_2 is a maximal element of the chain C_2') we have that $x_2' \preceq x_2$. Now, if $x_2 \preceq x_1$, then by the transitivity of $\preceq$, we have that $x_2' \preceq x_1$, which is impossible since x_1 and x_2' are distinct elements of the antichain A_1. So, $x_2 \not\preceq x_1$. An analogous argument establishes that $x_1 \not\preceq x_2$.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued).

> ***Claim 2.*** $\{x_1, \dots, x_k\}$ *is an antichain in* $(X_0, \preceq)$.

Proof of Claim 2. We may assume that $k \geq 2$, for otherwise, this is immediate. By symmetry, it suffices to show that x_1 and x_2 are incomparable. Let A_1 be an antichain of size k in $(X_0, \preceq)$ such that $x_1 \in A_1 \cap C_1$. By Claim 1, $|A_1 \cap C_2| = 1$; set $A_1 \cap C_2 = \{x_2'\}$. Then $x_2' \in C_2'$, and so (since x_2 is a maximal element of the chain C_2') we have that $x_2' \preceq x_2$. Now, if $x_2 \preceq x_1$, then by the transitivity of $\preceq$, we have that $x_2' \preceq x_1$, which is impossible since x_1 and x_2' are distinct elements of the antichain A_1. So, $x_2 \not\preceq x_1$. An analogous argument establishes that $x_1 \not\preceq x_2$. Thus, x_1 and x_2 are incomparable. This proves Claim 2.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued).

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Suppose first that $\{x_0, x_1, \ldots, x_k\}$ is an antichain in $(X, \preceq)$.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Suppose first that $\{x_0, x_1, \dots, x_k\}$ is an antichain in $(X, \preceq)$. Then this antichain is of size $k+1$, and $\{C_1, \dots, C_k, \{x_0\}\}$ is a chain decomposition of $(X, \preceq)$ of size $k+1$, and we are done.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Suppose first that $\{x_0, x_1, \ldots, x_k\}$ is an antichain in $(X, \preceq)$. Then this antichain is of size $k+1$, and $\{C_1, \ldots, C_k, \{x_0\}\}$ is a chain decomposition of $(X, \preceq)$ of size $k+1$, and we are done.

So, we may assume that $\{x_0, x_1, \ldots, x_k\}$ is not an antichain in $(X, \preceq)$.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Suppose first that $\{x_0, x_1, \dots, x_k\}$ is an antichain in $(X, \preceq)$. Then this antichain is of size $k + 1$, and $\{C_1, \dots, C_k, \{x_0\}\}$ is a chain decomposition of $(X, \preceq)$ of size $k + 1$, and we are done.

So, we may assume that $\{x_0, x_1, \dots, x_k\}$ is not an antichain in $(X, \preceq)$. By Claim 2, and by symmetry, we may assume that x_0 and x_1 are comparable;

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Suppose first that $\{x_0, x_1, \dots, x_k\}$ is an antichain in $(X, \preceq)$. Then this antichain is of size $k+1$, and $\{C_1, \dots, C_k, \{x_0\}\}$ is a chain decomposition of $(X, \preceq)$ of size $k+1$, and we are done.

So, we may assume that $\{x_0, x_1, \dots, x_k\}$ is not an antichain in $(X, \preceq)$. By Claim 2, and by symmetry, we may assume that x_0 and x_1 are comparable; since x_0 is a maximal element of $(X, \preceq)$, we see that $x_1 \preceq x_0$.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Suppose first that $\{x_0, x_1, \dots, x_k\}$ is an antichain in $(X, \preceq)$. Then this antichain is of size $k+1$, and $\{C_1, \dots, C_k, \{x_0\}\}$ is a chain decomposition of $(X, \preceq)$ of size $k+1$, and we are done.

So, we may assume that $\{x_0, x_1, \dots, x_k\}$ is not an antichain in $(X, \preceq)$. By Claim 2, and by symmetry, we may assume that x_0 and x_1 are comparable; since x_0 is a maximal element of $(X, \preceq)$, we see that $x_1 \preceq x_0$. Now, set $D_1 := \{x_0\} \cup \{x \in C_1 \mid x \preceq x_1\}$;

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Suppose first that $\{x_0, x_1, \dots, x_k\}$ is an antichain in $(X, \preceq)$. Then this antichain is of size $k+1$, and $\{C_1, \dots, C_k, \{x_0\}\}$ is a chain decomposition of $(X, \preceq)$ of size $k+1$, and we are done.

So, we may assume that $\{x_0, x_1, \dots, x_k\}$ is not an antichain in $(X, \preceq)$. By Claim 2, and by symmetry, we may assume that x_0 and x_1 are comparable; since x_0 is a maximal element of $(X, \preceq)$, we see that $x_1 \preceq x_0$. Now, set $D_1 := \{x_0\} \cup \{x \in C_1 \mid x \preceq x_1\}$; since C_1 is a chain, and $x_1 \preceq x_0$, the transitivity of $\preceq$ guarantees that D_1 is a chain in $(X, \preceq)$.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Reminder: $D_1 := \{x_0\} \cup \{x \in C_1 \mid x \preceq x_1\}$ is a chain.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Reminder: $D_1 := \{x_0\} \cup \{x \in C_1 \mid x \preceq x_1\}$ is a chain.

Further, by Claim 1, and by the choice of x_1, we know that $(X \setminus D_1, \preceq)$ does not have an antichain of size k.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Reminder: $D_1 := \{x_0\} \cup \{x \in C_1 \mid x \preceq x_1\}$ is a chain.

Further, by Claim 1, and by the choice of x_1, we know that $(X \setminus D_1, \preceq)$ does not have an antichain of size k. Since $\{x_2, \dots, x_k\}$ is an antichain of size $k-1$ in $(X \setminus D_1, \preceq)$, we deduce that the maximum size of an antichain in $(X \setminus D_1, \preceq)$ is $k-1$.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Reminder: $D_1 := \{x_0\} \cup \{x \in C_1 \mid x \preceq x_1\}$ is a chain.

Further, by Claim 1, and by the choice of x_1, we know that $(X \setminus D_1, \preceq)$ does not have an antichain of size k. Since $\{x_2, \dots, x_k\}$ is an antichain of size $k-1$ in $(X \setminus D_1, \preceq)$, we deduce that the maximum size of an antichain in $(X \setminus D_1, \preceq)$ is $k-1$. Then by the induction hypothesis, $(X \setminus D_1, \preceq)$ has a chain decomposition of size $k-1$, say $\{E_1, \dots, E_{k-1}\}$.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Proof (continued). Reminder: $D_1 := \{x_0\} \cup \{x \in C_1 \mid x \preceq x_1\}$ is a chain.

Further, by Claim 1, and by the choice of x_1, we know that $(X \setminus D_1, \preceq)$ does not have an antichain of size k. Since $\{x_2, \ldots, x_k\}$ is an antichain of size $k-1$ in $(X \setminus D_1, \preceq)$, we deduce that the maximum size of an antichain in $(X \setminus D_1, \preceq)$ is $k-1$. Then by the induction hypothesis, $(X \setminus D_1, \preceq)$ has a chain decomposition of size $k-1$, say $\{E_1, \ldots, E_{k-1}\}$. But now $\{D_1, E_1, \ldots, E_{k-1}\}$ is a chain decomposition of size k in $(X, \preceq)$, and we are done.

Definition

A *comparability graph* (or a *transitively orientable graph*) is a graph G such that there exists a partial order $\preceq$ on $V(G)$ such that for all distinct $x, y \in V(G)$, we have that $xy \in E(G)$ if and only if x and y are comparable with respect to $\preceq$.

Definition

A *comparability graph* (or a *transitively orientable graph*) is a graph G such that there exists a partial order $\preceq$ on $V(G)$ such that for all distinct $x, y \in V(G)$, we have that $xy \in E(G)$ if and only if x and y are comparable with respect to $\preceq$.

- Equivalently, G is a comparability graph if there exists an orientation $\vec{G} = (V(G), A(G))$ of G such that for all $\vec{uv}, \vec{vw} \in A(G)$, we have that $\vec{uw} \in A(G)$.

Definition

A *comparability graph* (or a *transitively orientable graph*) is a graph G such that there exists a partial order $\preceq$ on $V(G)$ such that for all distinct $x, y \in V(G)$, we have that $xy \in E(G)$ if and only if x and y are comparable with respect to $\preceq$.

- Equivalently, G is a comparability graph if there exists an orientation $\vec{G} = (V(G), A(G))$ of G such that for all $\vec{uv}, \vec{vw} \in A(G)$, we have that $\vec{uw} \in A(G)$.
- Note that in a comparability graph, cliques correspond to chains, and stable sets correspond to antichains.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Dilworth's theorem

In any finite partially ordered set $(X, \preceq)$, the maximum size of an antichain is equal to the minimum size of a chain decomposition of $(X, \preceq)$.

Corollary 2.1

Every comparability graph is perfect. The complement of any comparability graph is perfect.

Proof (outline). In view of the Perfect Graph Theorem, it suffices to show that the complement of any comparability graph is perfect. But this follows from Dilworth's theorem by an easy induction (details: Lecture Notes).

Lemma 3.1

Every bipartite graph is perfect.

Proof. Obvious.

Lemma 3.1

Every bipartite graph is perfect.

Proof. Obvious.

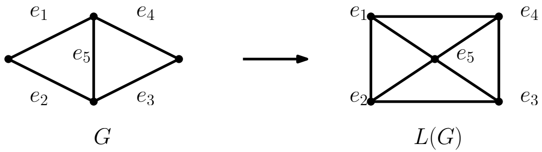
Lemma 3.2

The complement of any bipartite graph is perfect.

Proof. This follows immediately from Lemma 3.1 and the Perfect Graph Theorem.

Definition

Given a graph G, the *line graph* of G, denoted by $L(G)$, is the graph with vertex set $E(G)$, in which distinct $e, f \in E(G)$ are adjacent if and only if they share an endpoint in G.



Lemma 3.3

The line graph of any bipartite graph is perfect.

Proof (outline).

Lemma 3.3

The line graph of any bipartite graph is perfect.

Proof (outline). This essentially follows from the fact that every bipartite graph G satisfies $\chi'(G) = \Delta(G)$. (Details: Lecture Notes.)

Lemma 3.3

The line graph of any bipartite graph is perfect.

Proof (outline). This essentially follows from the fact that every bipartite graph G satisfies $\chi'(G) = \Delta(G)$. (Details: Lecture Notes.)

Lemma 3.4

The complement of the line graph of any bipartite graph is perfect.

Proof. This follows immediately from Lemma 3.3 and the Perfect Graph Theorem.

Definition

A *hole* in a graph G is an induced cycle of length at least four. An *antihole* in G is an induced subgraph H of G such that $\overline{H}$ is a hole in $\overline{G}$.

- Holes:

C_4 C_5 C_6 C_7

- Antiholes:

$\overline{C_4} \cong 2K_2$ $\overline{C_5} \cong C_5$ $\overline{C_6}$ $\overline{C_7}$

Definition

An *odd hole* (resp. *odd antihole*) is a hole (resp. antihole) that has an odd number of vertices.

Definition

An *odd hole* (resp. *odd antihole*) is a hole (resp. antihole) that has an odd number of vertices.

Definition

A graph is *Berge* if it contains no odd holes and no odd antiholes.

Definition

An *odd hole* (resp. *odd antihole*) is a hole (resp. antihole) that has an odd number of vertices.

Definition

A graph is *Berge* if it contains no odd holes and no odd antiholes.

The Strong Perfect Graph Theorem [Chudnovsky, Robertson, Seymour, Thomas, 2002]

A graph is perfect if and only if it is Berge.

Definition

An *odd hole* (resp. *odd antihole*) is a hole (resp. antihole) that has an odd number of vertices.

Definition

A graph is *Berge* if it contains no odd holes and no odd antiholes.

The Strong Perfect Graph Theorem [Chudnovsky, Robertson, Seymour, Thomas, 2002]

A graph is perfect if and only if it is Berge.

- Clearly, a graph is Berge if and only if its complement is Berge.

Definition

An *odd hole* (resp. *odd antihole*) is a hole (resp. antihole) that has an odd number of vertices.

Definition

A graph is *Berge* if it contains no odd holes and no odd antiholes.

The Strong Perfect Graph Theorem [Chudnovsky, Robertson, Seymour, Thomas, 2002]

A graph is perfect if and only if it is Berge.

- Clearly, a graph is Berge if and only if its complement is Berge.
- So, the Strong Perfect Graph Theorem immediately implies the Perfect Graph Theorem.

The Strong Perfect Graph Theorem [Chudnovsky, Robertson, Seymour, Thomas, 2002]

A graph is perfect if and only if it is Berge.

- One direction of the Strong Perfect Graph Theorem ("every perfect graph is Berge") is an easy exercise.

The Strong Perfect Graph Theorem [Chudnovsky, Robertson, Seymour, Thomas, 2002]

A graph is perfect if and only if it is Berge.

- One direction of the Strong Perfect Graph Theorem ("every perfect graph is Berge") is an easy exercise.
- Indeed, it is easy to check that for each integer $n \geq 2$, we have that
  - $\omega(C_{2n+1}) = 2$ and $\chi(C_{2n+1}) = 3$;
  - $\omega(\overline{C_{2n+1}}) = n$ and $\chi(\overline{C_{2n+1}}) = n + 1$.

The Strong Perfect Graph Theorem [Chudnovsky, Robertson, Seymour, Thomas, 2002]

A graph is perfect if and only if it is Berge.

- One direction of the Strong Perfect Graph Theorem ("every perfect graph is Berge") is an easy exercise.
- Indeed, it is easy to check that for each integer $n \geq 2$, we have that
  - $\omega(C_{2n+1}) = 2$ and $\chi(C_{2n+1}) = 3$;
  - $\omega(\overline{C_{2n+1}}) = n$ and $\chi(\overline{C_{2n+1}}) = n + 1$.
- So, odd holes and antiholes are imperfect, and therefore, no perfect graph contains an odd hole or an odd antihole. Thus, every perfect graph is Berge.

The Strong Perfect Graph Theorem [Chudnovsky, Robertson, Seymour, Thomas, 2002]

A graph is perfect if and only if it is Berge.

- How about the other direction ("every Berge graph is perfect")?

The Strong Perfect Graph Theorem [Chudnovsky, Robertson, Seymour, Thomas, 2002]

A graph is perfect if and only if it is Berge.

- How about the other direction ("every Berge graph is perfect")?
- It relies on a "decomposition theorem" for Berge graphs, which, roughly, states that every Berge graph either is "basic" or admits a "decomposition."

The Strong Perfect Graph Theorem [Chudnovsky, Robertson, Seymour, Thomas, 2002]

A graph is perfect if and only if it is Berge.

- How about the other direction ("every Berge graph is perfect")?
- It relies on a "decomposition theorem" for Berge graphs, which, roughly, states that every Berge graph either is "basic" or admits a "decomposition."
  - The proof of this decomposition theorem is by far the most complicated part of the proof of the Strong Perfect Graph Theorem, and it is over 100 pages long.

The Strong Perfect Graph Theorem [Chudnovsky, Robertson, Seymour, Thomas, 2002]

A graph is perfect if and only if it is Berge.

- How about the other direction ("every Berge graph is perfect")?
- It relies on a "decomposition theorem" for Berge graphs, which, roughly, states that every Berge graph either is "basic" or admits a "decomposition."
  - The proof of this decomposition theorem is by far the most complicated part of the proof of the Strong Perfect Graph Theorem, and it is over 100 pages long.
- The "basic" graphs are bipartite graphs and their complements, line graphs of bipartite graphs, complements of line graphs of bipartite graphs, and "double split" graphs (we omit the definition).

The Strong Perfect Graph Theorem [Chudnovsky, Robertson, Seymour, Thomas, 2002]

A graph is perfect if and only if it is Berge.

- How about the other direction ("every Berge graph is perfect")?
- It relies on a "decomposition theorem" for Berge graphs, which, roughly, states that every Berge graph either is "basic" or admits a "decomposition."
  - The proof of this decomposition theorem is by far the most complicated part of the proof of the Strong Perfect Graph Theorem, and it is over 100 pages long.
- The "basic" graphs are bipartite graphs and their complements, line graphs of bipartite graphs, complements of line graphs of bipartite graphs, and "double split" graphs (we omit the definition).
- All basic graphs are perfect: we proved this for the first four types of basic graphs, and the proof for double split graphs is easy.

The Strong Perfect Graph Theorem [Chudnovsky, Robertson, Seymour, Thomas, 2002]

A graph is perfect if and only if it is Berge.

- There are several “decompositions” (we omit the definitions), and it can be shown that no imperfect Berge graph of minimum possible size admits any of these decompositions.

The Strong Perfect Graph Theorem [Chudnovsky, Robertson, Seymour, Thomas, 2002]

A graph is perfect if and only if it is Berge.

- There are several “decompositions” (we omit the definitions), and it can be shown that no imperfect Berge graph of minimum possible size admits any of these decompositions.
- It now follows that all Berge graphs are perfect.

Algorithmic considerations:

Algorithmic considerations:

- Berge graphs can be recognized in $O(n^9)$ time (Chudnovsky, Cornuéjols, Liu, Seymour and Vušković 2005).

Algorithmic considerations:

- Berge graphs can be recognized in $O(n^9)$ time (Chudnovsky, Cornuéjols, Liu, Seymour and Vušković 2005).
    - By the Strong Perfect Graph Theorem, it follows that perfect graphs can be recognized in $O(n^9)$ time.

Algorithmic considerations:

- Berge graphs can be recognized in $O(n^9)$ time (Chudnovsky, Cornuéjols, Liu, Seymour and Vušković 2005).
  - By the Strong Perfect Graph Theorem, it follows that perfect graphs can be recognized in $O(n^9)$ time.
- Grötschel, Lovász, and Schrijver (1981) showed that the following optimization problems can be solved in polynomial time for perfect graphs: Maximum Clique, Maximum Stable Set, Graph Coloring (i.e. Vertex Coloring), and Minimum Clique Cover.

Algorithmic considerations:

- Berge graphs can be recognized in $O(n^9)$ time (Chudnovsky, Cornuéjols, Liu, Seymour and Vušković 2005).
  - By the Strong Perfect Graph Theorem, it follows that perfect graphs can be recognized in $O(n^9)$ time.
- Grötschel, Lovász, and Schrijver (1981) showed that the following optimization problems can be solved in polynomial time for perfect graphs: Maximum Clique, Maximum Stable Set, Graph Coloring (i.e. Vertex Coloring), and Minimum Clique Cover.
  - In fact, weighted versions of these problems can also be solved in polynomial time.